Preparatory Class Lessons

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PART I

Maths

1 Logic

1.1 Logical Propositions

Definition

A logical proposition p is a mathematical phrase that is either true or false.

The negation of a proposition is written $\neg$: $\neg p$ reads "not p"
A logical operator can be defined by its truth table :
$p \Rightarrow q$ is defined by $\neg p$ or q, or by the following table (T is short for True and F is short for False) :

p	q	$\neg p$	$\neg p$ or q
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

$p \Rightarrow q$ reads "if p then q" or "p implies q". Notice that is it always true if p is wrong!

1.2 Quantifiers

$\forall$ reads "for all" or "for each" or "for every".
$\exists$ reads "exists".
$\exists!$ reads "exists a unique" or "exists an only"

Negating a quantified proposition

A quantified proposition will always look like this "Quantifier, Quantifier, ... , Quantifier, p" where p is a proposition with no quantifiers.
In order to negate that proposition, you negate every quantifier and p.
The opposite of $\forall$ is $\exists$ (and conversely).

Ex 1

Let $A, B \subset \mathbb{C}$.
Negate $p =$ "$\forall a \in A, \quad \exists b \in B, a + b = 0$".

1.3 Reasoning techniques

1.3.1 Equivalence

$p \Leftrightarrow q$ reads "p if and only if q" or "p is equivalent to q". The equivalence method (or reasoning "by equivalence") is when you prove something via a chain of equivalences. This is the shortest and most difficult way to prove a proposition of the form "$p \Leftrightarrow q$". Another way of proving $p \Leftrightarrow q$ is to separate the proofs of $p \Rightarrow q$ and $q \Rightarrow p$.

Ex 2

For $n \in \mathbb{N}$, we define $z_n := (1 + i\sqrt{3})^n$.
Find all the $n \in \mathbb{N}$ that satisfy $z_n \in \mathbb{R}_+$

1.3.2 Analysis-Synthesis

The Analysis-Synthesis method

A/S is used to answer questions like "find all the x that satisfy ..." or "prove that there exists a unique x that satisfies ...".
1) *Analysis* : you analyse a solution and discover its properties : "Let x be a solution, then x satisfies ... so ... so ...". At the end you want $x \in S'$ where S' is a small set you hope is the solution set. In the analysis you draw **necessary** conditions.
2) *Synthesis* : you check that S' is indeed the solution set, or you find conditions on its elements to be solutions : you end up with a subset of S' that is the solution set. In the synthesis you highlight **sufficient** conditions.

Ex 3

Find all the functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ that are differentiable and that satisfy the equation :
$\forall (x, y) \in \mathbb{R}^2, \quad f(x + y) = f(x) + f(y)$

Ex 4

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ a function.
Show that $\exists!(a, b) \in F(\mathbb{R}, \mathbb{R})^2$ so that $f = a + b$ and a is even and b is odd.

1.3.3 Contraposition

The Contraposition method

$p \Rightarrow q$ has the same value as $\neg q \Rightarrow \neg p$.

Ex 5

Let $n \in \mathbb{N}$.
Prove that "n^2 is odd" $\Rightarrow$ "n is odd".

Ex 6

Let $a \in \mathbb{R}$. Show by contraposition that :
"$\forall \varepsilon > 0, \quad |a| \leq \varepsilon$ " $\Rightarrow \quad a = 0$.

1.3.4 Proof by contradiction/ "by the absurd"

The contradiction method

In order to show that a proposition p is true, you can prove that $\neg p$ implies something contradictory. If $\neg p$ is absurd, then p is true.

Ex 7

Prove the unicity of the limit of a convergent sequence by supposing it has two different limits.

1.4 For next time

Ex 8

Prove the contraposition method by proving that $p \Rightarrow q$ and $\neg q \Rightarrow \neg p$ have the same values.
To do that you can either draw both truth tables for $p \Rightarrow q$ and $\neg q \Rightarrow \neg p$, or write their definitions.

Ex 9

Find the functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ that satisfy :
$\forall (x, y) \in \mathbb{R}^2, \quad f(x) \times f(y) - f(x \times y) = x + y$
Hint : find $f(0)$.

Ex 10

Show that $\sqrt{2}$ is irrational.

(And irrational number is a number x that cannot be written in the form $x = \dfrac{p}{q}$ where $p \in \mathbb{Z}$ and $q \in \mathbb{N}^*$).

Learn the following content :

Definition

For all $n \in \mathbb{N}$, we define $n!$ (pronounce "factorial n") by $n! = 1 \times 2 \times ... \times n$ and $0! = 1$.

For all $(k, n) \in \mathbb{N}^2$, we define $\dbinom{n}{k}$ (read "n choose k") by $\dfrac{n!}{k!(n-k)!}$.

Remark : $\dbinom{n}{k}$ is the number of possibilities of choosing k objects within n objects.

Binomial properties

Let $(a, b) \in \mathbb{N}^2$. We have :

$$\binom{b}{a} = \binom{b}{b-a} \quad b < a \Rightarrow \binom{b}{a} = 0, \quad \binom{b}{0} = 0, \quad \binom{b}{1} = b$$

Pascal's formula : if $a, b \geq 1$, $\dbinom{b-1}{a-1} + \dbinom{b-1}{a} = \dbinom{b}{a}$

Binomial theorem : $\forall (x, y) \in \mathbb{C}^2, \quad \forall n \in \mathbb{N}, \quad (x+y)^n = \displaystyle\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$

Ex 11

Let $n \in \mathbb{N}$. Compute the following quantity :

$$\sum_{k=0}^{n} \binom{n}{k}$$

1.5 Homework Correction

1.5.1 Correction of Ex 8

We shall write the symbol $\equiv$ to express that two propositions have the same value.

$$(\neg q \Rightarrow \neg p) \equiv (\neg(\neg q) \text{ or } \neg p) \equiv (q \text{ or } \neg p) \equiv (p \Rightarrow q)$$

1.5.2 Correction of Ex 9

Analysis

Let f be a solution to the equation.

In particular, by applying the equation at $(x, y) = (0, 0)$, we have $f(0)^2 - f(0) = 0$, and thus $f(0) = 0$ or $f(0) = 1$.

Let $x \in \mathbb{R}$. By applying the equation at $(x, 0)$ we have :

$f(x) \times f(0) - f(0) = x$.

We deduce that $f(0) \neq 0$ otherwise at $x = 1$ we would have $0 = 1$. Therefore $f(0) = 1$.

Conclusion of the Analysis : $\forall x \in \mathbb{R}, \quad f(x) = x + 1$.

Synthesis

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $\forall x \in \mathbb{R}, \quad f(x) = x + 1$.

Let $(x, y) \in \mathbb{R}^2$. $f(x)f(y) - f(xy) = (x + 1)(y + 1) - xy - 1 = x + y$.

Therefore f is a solution of the equation.

Finally, the only solution of the equation is $\boxed{x \longmapsto x + 1}$.

1.5.3 Correction of Ex 10

Let us reason by contradiction : we suppose that there exists $(p, q) \in \mathbb{N} \times \mathbb{N}^*$ so that $\sqrt{2} = \frac{p}{q}$.

We thus have $2 = \frac{p^2}{q^2}$, therefore $p^2 = 2q^2$.

Let us write the decomposition into prime factors of a number $a \in \mathbb{N}$: $a = p_1^{\alpha_1} \times p_2^{\alpha_2} \times ... \times p_n^{\alpha_n}$, where $p_1, ..., p_n$ are distinct prime numbers, $\alpha_1, ..., \alpha_n \in \mathbb{N}^*$ and $n \in \mathbb{N}$. (for instance $24 = 2^3 \times 3$.)

Notice that $a^2 = p_1^{2\alpha_1} \times ... \times p_n^{2\alpha_n}$, hence for each prime number P, the maximum amount of times you can divide a^2 by P is even : either P is one of the p_i thus that amount is $2\alpha_i$ which is even, or P is different to every p_i and that amount is 0, which is also even.

Let us define m_1 the maximum amount of times you can divise p by 2 and m_2 the same for q. The maximum amount of times you can divide p^2 by 2 is $2m_1$ (respectively $2m_2$ for q^2).

Since $p^2 = 2q^2$, we have $2m_1 = 1 + 2m_2$ (you can divide $2q^2$ $\;1 + 2m_2$ times by 2).

That equation is absurd because on the left side you have an even number and on the left you have an odd one.

Conclusion : the supposition "$\exists (p, q) \in \mathbb{N} \times \mathbb{N}^*, \quad \sqrt{2} = \frac{p}{q}$" is absurd and so $\boxed{\sqrt{2} \text{ is irrational.}}$

1.6 Correction of Ex 11

We use the Binomial theorem with $(x, y) = (1, 1)$: $\sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k} 1^k \times 1^{n-k} = (1 + 1)^n = \boxed{2^n}$

2 Induction

Note : induction can also be called recursion.

2.1 Simple Induction

Definition

Induction is a way of proving a property P_n that depends on a natural number n.

Proving by (simple) induction is saying that :
if the property is true at the rank $n = 0$
and that for all n, if the property is true at n then it is true at the rank $n + 1$,
then it is true for all n.

Therefore, proof by induction is always done in two steps :
1) Initialisation Prove for $n = 0$
2) Induction Let $n \in \mathbb{N}$. Suppose the property true at rank n (suppose P_n true). Using that, prove P_{n+1}.

Remarks :

- For the induction phase, you can also go from $n - 1$ to n, in that case you must suppose $n \geq 1$.
- Using the same principle, induction can also define objects ("by induction") For example, one can define $n!$ for all $n \in \mathbb{N}$ by $0! = 1$ and $\forall n \geq 1, \quad n! = n \times (n-1)!$
- You can do a "finite induction" by using an induction to prove a property for $0 \leq n \leq M$ instead of for all $n \in \mathbb{N}$. The process is exactly the same, you just have to suppose n smaller than $M - 1$ when you prove P_{n+1} with P_n.

Ex 12

Let $q \in \mathbb{C} \setminus \{1\}$ and $n \in \mathbb{N}$.
Show that $\displaystyle\sum_{k=0}^{n} q^k = \frac{1 - q^{n+1}}{1 - q}$

Ex 13

$$\forall n \in \mathbb{N}, \quad \sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Ex 14

Let $(a, b) \in \mathbb{C}^2$ and $n \in \mathbb{N}$.
Prove the Binomial theorem : $\displaystyle(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$

2.2 Strong Induction

Definition

Strong induction is the same as simple induction, except that during the induction phase, instead of supposing the previous step true, you suppose **all** previous steps true.

1) Initialisation Prove for $n = 0$
2) Induction Let $n \in \mathbb{N}$, suppose that $\forall k \in [\![0, n]\!]$, P_k is true, and prove P_{n+1}.

You can also suppose $P_0, ..., P_{n-1}$ to be true and prove P_n if you give yourself $n \geq 1$.

Ex 15

Let (u_n) be defined by : $u_0 = 1$, $\forall n \in \mathbb{N}, \quad u_{n+1} = u_0 + ... + u_n$
Prove that $\forall n \geq 1, \quad u_n = 2^{n-1}$

Ex 16

Euclidian division.
Let $(a, b) \in \mathbb{N} \times \mathbb{N}^*$. Prove that $\exists!(q, r) \in \mathbb{N} \times [\![0, b-1]\!], \quad a = bq + r$.

2.3 Exercises

Ex 17

Let A be a subset of $\mathbb{N}^*$ that satisfies the properties :

- $1 \in A$
- $\forall n \in A, \quad 2n \in A$
- $\forall n \in \mathbb{N}^*, \quad n+1 \in A \Rightarrow n \in A$

Prove that $A = \mathbb{N}^*$
Hint : try to prove $n \in A$ for small values of n.

Ex 18

Let $a < b$ two real numbers. Let $f : [a, b] \longrightarrow [a, b]$ be K-Lipschitz continuous function with $0 < K < 1$.
Reminder : f is K-Lipschitz continous means that $\forall (x, y) \in [a, b]^2, \quad |f(x) - f(y)| \leq K|x - y|$
Show that the sequence (u_n) defined by $u_0 \in [a, b]$, $\forall n \in \mathbb{N}, \quad u_{n+1} = f(u_n)$ converges towards a fixed point p of f (p exists thanks to the Intermediate Values Theorem).

2.4 Sets and maps

Set operators

Let A and B be two sets.

The belonging of an element to a set is written $\in$, (read "in") : $a \in A$ means that the element a belongs to the set A.

The inclusion $A \subset B$ is a proposition that means $\forall a \in A, \quad a \in B$.

The equality $A = B$ means that $A \subset B$ **and** $B \subset A$. Seperating the two $\subset$ is a useful way to prove that two sets are equal.

Sets can be defined by two methods : by "direct image" (for instance let $E = \{x^2 + 1 | x \in \mathbb{R})$ or by "conditions" (for instance $\{x \in \mathbb{R} | x^2 + x + 1 = 0\}$.)

The privation $\setminus$ substracts a set from another : $[0, 2] \setminus]1, 2] = [0, 1]$.

The union $A \cup B$ is the set of the elements that are in A or in B. The intersection $A \cap B$ is the set composed of elements that are in **both** A and B.

When defining several n objects in A, you must write "Let $(a_1, ..., a_n) \in A^n$.

The cartesian product $A \times B$ is defined by $A \times B = \{(a, b) | a \in A, b \in B\}$.

Maps

Let A, B be two sets. A map from A to B is defined in the following manner :

$$f : \begin{cases} A & \longrightarrow & B \\ x & \longmapsto & f(x) \end{cases}$$ Where $f(x)$ has an explicit expression.

A is the map's domain is B its target. $f(A)$ is f's image.

Let $b \in B$ and $a \in A$. If $f(a) = b$ then a is called a **fiber** of b.

An **injection** is a map satisfying $\forall (a_1, a_2) \in A^2, \quad f(a_1) = f(a_2) \Rightarrow a_1 = a_2$. An injection can only have one or zero fibers per $b \in B$

A **surjection** is a map satisfying $\forall b \in B, \quad \exists a \in A, \quad f(a) = b$. Every $b \in B$ has at least one fiber by f.

A **bijection** or **one-to-one** map is a map that is both injective and surjective. It satisfies $\forall b \in B, \quad \exists! a \in A, \quad f(a) = b$. Each $b \in B$ has one and only one fiber by a bijection.

3 Complex Numbers

3.1 Definitions

The set of complex numbers $\mathbb{C}$ is the set $\{x + iy | (x, y) \in \mathbb{R}^2\}$. "$i$" is a quantity that satisfies $i^2 = -1$. Warning, do not write $\sqrt{-1}$ because that has no meaning! $(-i)^2 = -1$ too.

Complex exponential

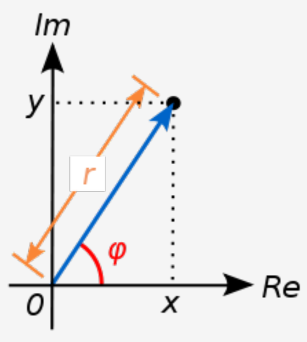
$$\forall \varphi \in \mathbb{R}, \quad e^{i\varphi} = \cos(\varphi) + i \sin(\varphi)$$

.

Representations

Algebric representation :
$\forall z \in \mathbb{C}, \quad \exists! (x, y) \in \mathbb{R}^2, \quad z = x + iy$. We define $\mathrm{Re}(z) := x$ (real part) and $\mathrm{Im}(z) := y$ (imaginary part).

Polar representation :
$\forall z \in \mathbb{C}, \quad \exists! (r, \varphi) \in \mathbb{R}_+ \times [0, 2\pi[, \quad z = re^{i\varphi}$. We define $|z| := r$ (modulus) and $\mathrm{Arg}(z) := \varphi$ (its **primary** argument).



If $z = a + ib \in \mathbb{C}$, we have $|z| = \sqrt{a^2 + b^2}$

Definition

The conjugate of $z = a + ib \in \mathbb{C}$ is $\overline{z} = a - ib$.
We also have $\forall \varphi \in \mathbb{R}, \quad \overline{e^{i\varphi}} = e^{-i\varphi}$

Ex 19

Let $z \in \mathbb{C}$. Prove :

$$\mathrm{Re}(z) = \frac{z + \overline{z}}{2} \quad \text{and} \quad \mathrm{Im}(z) = \frac{z - \overline{z}}{2i}.$$

Corollary

Let $x \in \mathbb{R}$. Applying the previous exercise to e^{ix} gives the EULER formula :
$\cos(x) = \dfrac{e^{ix} + e^{-ix}}{2}$ and $\sin(x) = \dfrac{e^{ix} - e^{-ix}}{2i}$
Using the Binomial Theorem (BT), we can linearise $\cos^n(x)$ and $\sin^n(x)$ (transform them into sums of $\cos(kx)$ and $\sin(kx)$)

Linearisation of $\cos^3$: Let $x \in \mathbb{R}$.

$$\begin{aligned}
\cos^3(x) &= \left(\frac{e^{ix} + e^{-ix}}{2}\right)^3 \quad (Euler) \\
&= 2^{-3}\left(e^{3ix} + 3e^{ix} + 3e^{-ix} + e^{-3ix}\right) \quad (BT) \\
&= 2^{-2}\left(\cos(3x) + 3\cos(x)\right) \quad (Euler) \\
&= \frac{1}{4}\cos(3x) + \frac{3}{4}\cos(x)
\end{aligned}$$

Ex 20

Let $x \in R$.
Linearise $\sin^4(x)$.

Properties of the conjugate, the modulus and the argument

Let $(z, z') \in \mathbb{C}^2$. $\overline{z + z'} = \overline{z} + \overline{z'}$ $\overline{zz'} = \overline{z} \times \overline{z'}$.

Triangular inequality : $|z + z'| \leq |z| + |z'|$

Second triangular inequality : $||z| - |z'|| \leq |z - z'|$

Suppose z and z' nonzero. $|z + z'| = |z| + |z'| \quad \Leftrightarrow \quad \exists \lambda \in \mathbb{R}_+, \quad z = \lambda z'$

$|zz'| = |z| \times |z'|$

$\operatorname{Arg}(zz') \equiv \operatorname{Arg}(z) + \operatorname{Arg}(z')[2\pi]$

Complex exponential

exp can be continued to $\mathbb{C}$ with the formula : $\forall z = a + ib \in \mathbb{C}, \quad e^z := e^a \times e^{ib}$

We have $\forall (z, z') \in \mathbb{C}^2, \quad e^{z+z'} = e^z \times e^{z'}$ and $e^{\overline{z}} = \overline{e^z}$

$\forall \theta \in \mathbb{R}, \quad |e^{i\theta}| = 1$ And thus $\cos^2\theta + \sin^2\theta = 1$

Ex 21

Let $z = a + ib \in \mathbb{C}$.
Compute $|e^z|$.

Trigonometric formulas

Let $(a,b) \in \mathbb{R}^2$.
$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$, $\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$
$\sin(a+b) = \sin(a)\cos(b) + \sin(b)\cos(a)$, $\sin(a-b) = \sin(a)\cos(b) - \sin(b)\cos(a)$

$\cos(2a) = \cos^2(a) - \sin^2(a) = 2\cos^2(a) - 1 = 1 - 2\sin^2(a)$
$\sin(2a) = 2\cos(a)\sin(a)$

$$\cos^2(a) = \frac{1+\cos(2a}{2}, \quad \sin^2(a) = \frac{1-\cos(2a)}{2}$$

$\cos(a)\cos(b) = \frac{1}{2}(\cos(a+b) + \cos(a-b))$
$\sin(a)\sin(b) = -\frac{1}{2}(\cos(a+b) - \cos(a-b))$
$\sin(a)\cos(b) = \frac{1}{2}(\sin(a+b) + \sin(a-b))$

$\cos(a) + \cos(b) = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$, $\cos(a) - \cos(b) = -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$
$\sin(a) + \sin(b) = 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$, $\sin(a) - \sin(b) = 2\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$

If you have any doubts, use the parity of cos and sin and their particular values to check you formulas.

Expansion

Let $x \in \mathbb{R}$ and $n \in \mathbb{N}$. The Moivre formula $\boxed{\cos(nx) + i\sin(nx) = (\cos x + i\sin x)^n}$ allows to express $\cos(nx)$ or $\sin(nx)$ as polynomials in $\cos(x)$ and $\sin(x)$ using the Binomial Theorem.

To do that you write $\cos(nx) = \mathrm{Re}\left((\cos(x) + i\sin(x))^n\right)$ or $\sin(nx) = \mathrm{Im}\left(\cos(x) + i\sin(x))^n\right)$, then you use the Binomial Theorem to expand the power.

Ex 22

Let $\theta \in \mathbb{R}$.
Expand $\cos(3\theta)$ into a polynomial in $\cos(\theta)$ and $\sin(\theta)$

Ex 23

Let $x \in \mathbb{R}$ and $n \in \mathbb{N}$.
Compute $S_1(x) = \sum_{k=0}^{n} \cos(kx)$ and $S_2(x) = \sum_{k=0}^{n} \binom{n}{k} \sin(kx)$

Formulas around π

Let $a \in \mathbb{R}$.
$\cos(-a) = \cos(a)$, $\sin(-a) = -\sin(a)$

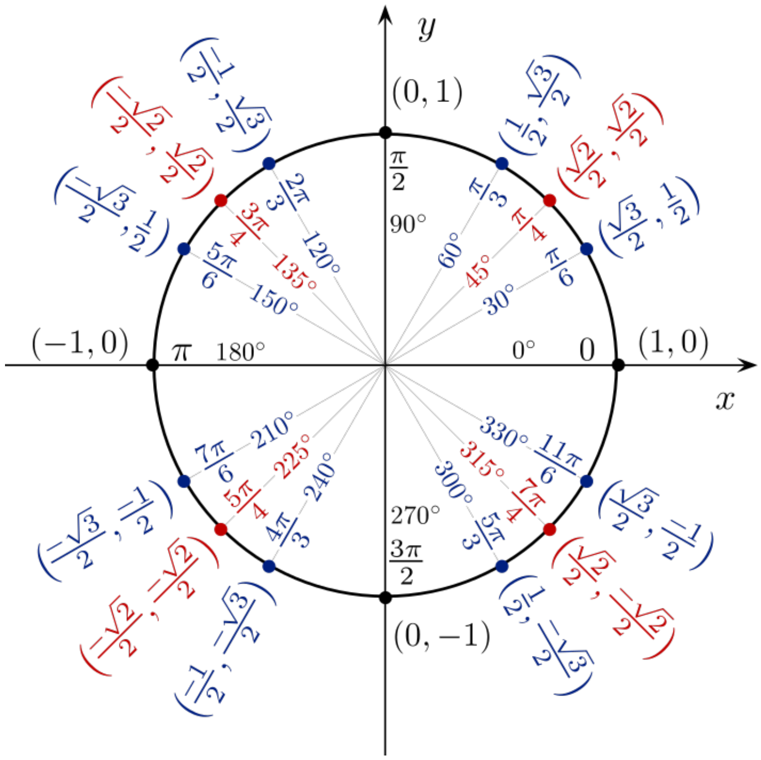
$\cos(\pi - a) = -\cos(a)$, $\sin(\pi - a) = \sin(a)$

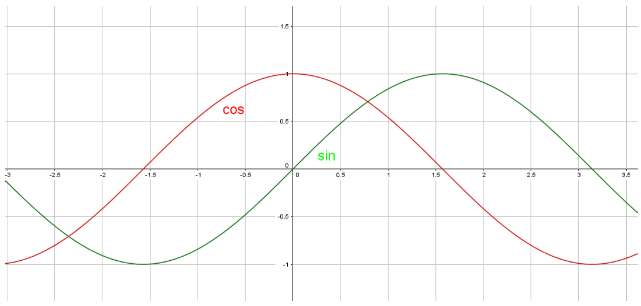
$\cos(\pi + a) = -\cos(a)$, $\sin(\pi + a) = -\sin(a)$

$\cos(\frac{\pi}{2} - a) = \sin(a)$, $\sin(\frac{\pi}{2} - a) = \cos(a)$

$\cos(\frac{\pi}{2} + a) = -\sin(a)$, $\sin(\frac{\pi}{2} + a) = \cos(a)$

To memorise this, use the trigonometric circle (cos is the projection on the horizontal x axis and sin the projection on the vertical y axis). You also need to know some particular values (summarised in this diagram) :





Half-arc formulas

Let $(a,b) \in \mathbb{R}^2$.
$e^{ia} + 1 = 2e^{i\frac{a}{2}} \cos \frac{a}{2}$, $\quad e^{ia} - 1 = 2ie^{i\frac{a}{2}} \sin \frac{a}{2}$

$e^{ia} + e^{ib} = 2e^{i\frac{a+b}{2}} \cos \frac{b-a}{2}$, $\quad e^{ia} - e^{ib} = 2ie^{i\frac{a+b}{2}} \sin \frac{a-b}{2}$

Ex 24

Prove the previous equations.
Hint : use EULER*'s formula).*

Definition

Let $x \in \mathbb{R}$ that satisfies $x \not\equiv \dfrac{\pi}{2}[\pi]$, we define its **tangent** $\tan(x) = \dfrac{\cos(x)}{\sin x}$.

Let $a, b \in \mathbb{R} \setminus (\frac{\pi}{2} + \pi\mathbb{Z})$. We have : $\tan(a+b) = \dfrac{\tan(a) + \tan(b)}{1 - \tan(a)\tan(b)}$

Ex 25

Let $a \in \mathbb{R} \setminus (\frac{\pi}{2} + \pi\mathbb{Z})$.
Prove that $\tan(a)\tan(\frac{\pi}{2} - a) = 1$.

Half-tangent formulas

Let $\theta \not\equiv \pi[2\pi]$. Let $t := \tan\frac{\theta}{2}$. We have

$$\cos\theta = \frac{1-t^2}{1+t^2}, \quad \sin\theta = \frac{2t}{1+t^2}, \quad \tan\theta = \frac{2t}{1-t^2}$$

To check, use that cos is even and that sin and tan are odd. You can also use that tan isn't always defined, but the others are.

Proof

1) $\cos\theta = 2\cos^2\frac{\theta}{2} - 1 = \dfrac{1}{1+t^2} - 1$ (since $\dfrac{1}{\cos^2\frac{\theta}{2}} = 1+\tan^2\frac{\theta}{2}$.)

Finally $\boxed{\cos\theta = \dfrac{1-t^2}{1+t^2}}$

2) $\tan\theta = \tan(\frac{\theta}{2}+\frac{\theta}{2}) = \boxed{\dfrac{2t}{1-t^2}}$

3) $\sin\theta = \tan\theta\cos\theta = \dfrac{2t}{1-t^2} \times \dfrac{1-t^2}{1+t^2} = \boxed{\dfrac{2t}{1+t^2}}$

Square roots of unity

Let $n \in \mathbb{N}^*$. The equation $z^n = 1$ has exactly n solutions which are the $e^{\frac{2ik\pi}{n}}$ with $k \in [\![0, n-1]\!]$. The are called the n-th roots of unity, we write their set $\mathbb{U}_n$.

Proof

Analysis
Let $z \in \mathbb{C}$ be a n-th root of unity. Let us write $z = re^{i\theta}$ its polar expression.

We have, since $z^n = 1$, $r^n = 1$, therefore $r = 1$ and $n\theta \equiv 0[2\pi]$, hence $\exists k \in \mathbb{Z}, \quad n\theta = 2k\pi$ thus $\exists k \in \mathbb{Z}, \quad \theta = \dfrac{2k\pi}{n}$

Yet $\theta \in [0, 2\pi[$ thus $k \in [\![0, n-1]\!]$.

Synthesis
Let $k \in [\![0, n-1]\!]$.

We have $\left(e^{\frac{2ik\pi}{n}}\right)^n = e^{2ik\pi} = 1$.

n solutions ?

We now have to prove that the $e^{\frac{2ik\pi}{n}}$ are distinct two by two. Let $(k, k') \in [\![0, n-1]\!]^2$.

Suppose $e^{\frac{2ik\pi}{n}} = e^{\frac{2ik'\pi}{n}}$. Since $0 \leq k, k' < n$, $0 \leq \frac{2ik\pi}{n}, \frac{2ik'\pi}{n} < 2\pi$.

By the unicity of the primary argument (both arguments are in $[0, 2\pi[$), we conclude that $k = k'$.

We thus have n distinct solutions.

Remark : Let $\omega := e^{\frac{2i\pi}{n}}$. We have $\mathbb{U}_n = \{\omega^k | k \in [\![0, n-1]\!]\}$

Ex 26

Let $n \in \mathbb{N}^*$. Compute $\displaystyle\sum_{z \in \mathbb{U}_n} z$.

n-th root of any complex number

Let $z \in \mathbb{C}^*$ and $n \in \mathbb{N}^*$.

Let $\omega := e^{\frac{2i\pi}{n}}$ and s be a particular n-th root of z. The n-th roots of z are the $\boxed{u\omega^k}$ for $k \in [\![0, n-1]\!]$.

Proof

Let $r \in \mathbb{C}$.

$$\begin{aligned} r \text{ is a n} - \text{th root of } z &\Leftrightarrow r^n = z \\ &\Leftrightarrow r^n = u^n \quad (u \text{ is a particular root}) \\ &\Leftrightarrow \left(\frac{r}{u}\right)^n = 1 \\ &\Leftrightarrow \frac{r}{u} \in \mathbb{U}_n \\ &\Leftrightarrow \exists k \in [\![0, n-1]\!], \quad r = u\omega^k \end{aligned}$$

How to find a particular n-th root

Let $z = re^{i\theta} \in \mathbb{C}^*$. $\boxed{u := \sqrt[n]{r}e^{i\frac{\theta}{n}}}$ is a particular n-th root of z.

How to find a square root of $z = a + ib$

If you only have $z \in \mathbb{C}$ in its algebric form $z = a+ib$, you can find its square roots by Analysis-Synthesis following this method :

Analysis

Let $r = x + iy$ be a square root of z.

Necessarily, $|r|^2 = |z|$ so if we have $x^2 + y^2 = |z|$.

Necessarily, $(x + iy)^2 = a + ib$ thus by applying Re and Im we have the system (*) :

$$\begin{cases} x^2 - y^2 &= a \\ 2xy &= b \end{cases}$$

In particular with the previous equation we have :

$$\begin{cases} x^2 - y^2 &= a \\ x^2 + y^2 &= |z| \end{cases} \text{ This solves in : } \begin{cases} x^2 &= \dfrac{|z| + a}{2} \\ y^2 &= \dfrac{|z| - a}{2} \end{cases}$$

Notice that $|z| > |a|$ thanks to the triangular inequality, so necessarily, $x = \pm\sqrt{\dfrac{|z| + a}{2}}$ and $y = \pm\sqrt{\dfrac{|z| - a}{2}}$

The second line of the system (*) read $2xy = b$. Therefore, there are two possibilities for the pair (x, y), that satisfy $\text{sign}(xy) = \text{sign}(b)$.

Synthesis

By theorem there are exactly two solutions, and we have found 2. Therefore, the two aforementioned solutions are the two only solutions.

Ex 27

Let $\Delta = 1 + 2i$.
Find all the square roots of Δ using the previous method.

Finding the roots of a second-degree complex polynomial

Let $P = aX^2 + bX + c$ with $(a, b, c) \in \mathbb{C}^* \times \mathbb{C} \times \mathbb{C}$.

Let $\Delta := b^2 - 4ac$, and δ be **a** square root of Δ.

1) If $\Delta \neq 0$, then there are two solutions $z_1 = \dfrac{-b+\delta}{2a}$ and $z_2 = \dfrac{-b-\delta}{2}$.

2) If $\Delta = 0$, then there is one solution $z_0 = -\dfrac{b}{2a}$

Ex 28

Prove that for all $z \in \mathbb{C}$, $P(z) = 0 \Leftrightarrow \left(z + \dfrac{b}{2a}\right)^2 = \dfrac{\Delta}{4a^2}$.

Using that equation, prove the previous theorem.

3.2 Homework Correction

3.2.1 Correction of Ex 8

Let $z = a + ib \in \mathbb{C}$. We have $\dfrac{z+\overline{z}}{2} = \dfrac{a+ib+a-ib}{2} = a$, and $\dfrac{z-\overline{z}}{2i} = \dfrac{a+ib-a+ib}{2i} = b$

3.2.2 Correction of Ex 9

$$\sin^4(x) = (2i)^{-4}\left(e^{ix} - e^{-ix}\right)^4 = \frac{1}{16}\left(e^{4ix} - 4e^{2ix} + 6e^{i0x} - 4e^{-2ix} + e^{-4ix}\right) = \boxed{\frac{1}{8}\cos(4x) - \frac{1}{2}\cos(2x) + \frac{3}{8}}$$

3.2.3 Correction of Ex 10

$$|e^z| = |e^{a+ib}| = |e^a| \times |e^{ib}| = \boxed{e^a}$$

3.2.4 Correction of Ex 11

$$\cos(3\theta) = \mathrm{Re}\left((\cos\theta + i\sin\theta)^3\right) = \mathrm{Re}\left(\cos^3\theta + 3i\cos^2\theta\sin\theta + 3i^2\cos\theta\sin^2\theta + i^3\sin^2\theta\right) = \boxed{\cos^3\theta - 3\cos\theta\sin^2\theta}$$

3.2.5 Correction of Ex 12

Suppose $x \not\equiv 0[2\pi]$. We have $S_1(x) = \mathrm{Re}\left(\displaystyle\sum_{k=0}^{n}(e^{ix})^k\right) = \mathrm{Re}\left(\dfrac{1-e^{i(n+1)x}}{1-e^{ix}}\right) = \mathrm{Re}\left(\dfrac{(1-e^{i(n+1)x})(1-e^{-ix})}{(1-\cos x)^2 + \sin^2 x}\right)$.

Therefore $S_1(x) = \mathrm{Re}\left(\dfrac{1 - e^{-ix} - e^{i(n+1)x} + e^{inx}}{2 - 2\cos x}\right) = \dfrac{1 - \cos(x) - \cos((n+1)x) + \cos(nx)}{2 - 2\cos(x)}$

Finally $\boxed{S_1(x) = \dfrac{1}{2} + \dfrac{\cos(nx) - \cos((n+1)x)}{2 - 2\cos(x)}}$. If $x \equiv 0[2\pi]$ then $S_1(x) = n$.

We have $S_2(x) = \mathrm{Im}\left(\displaystyle\sum_{k=0}^{n}\binom{n}{k}e^{ikx}\right) = \mathrm{Im}\left((1+e^{ix})^n\right) = \mathrm{Im}\left((2e^{i\frac{x}{2}}\cos(\tfrac{x}{2}))^n\right) = \boxed{2^n\sin(\tfrac{nx}{2})\cos^n(\tfrac{x}{2})}$

3.2.6 Correction of Ex 13

Let us prove the third equation first. We have $2e^{i\frac{a+b}{2}}\cos\frac{a+b}{2} = 2e^{i\frac{a+b}{2}}\dfrac{e^{i\frac{a+b}{2}}+e^{-i\frac{a+b}{2}}}{2} = e^{ia}+e^{ib}$. This gives the first equation with $b=0$. The last equation is proved in the same way, and it implies the second too.

3.2.7 Correction of Ex 14

$\tan(a)\tan(\frac{\pi}{2}-a) = \dfrac{\sin(a)\sin(\frac{\pi}{2}-a)}{\cos(a)\cos(\frac{\pi}{2}-a)} = \dfrac{\sin a\cos a}{\cos a\sin a} = 1$

3.2.8 Correction of Ex 15

$\displaystyle\sum_{z\in\mathbb{U}_n} z = \sum_{k=0}^{n-1}\omega^k$ where $\omega = e^{\frac{2i\pi}{n}}$ $(\neq 1)$. Therefore the sum equals to $\dfrac{1-\omega^n}{1-\omega} = \boxed{0}$.

3.2.9 Correction of Ex 16

Analysis. Let $\delta = x+iy$ be a square root of Δ.
Necessarily, $\delta^2=\Delta$ and $|\delta|^2 = |\Delta|$ therefore :

$$\begin{cases} x^2+y^2 &= |\Delta| \\ x^2-y^2 &= 1 \\ 2xy &= 2 \end{cases}$$

Therefore $x = \pm\sqrt{\dfrac{\sqrt{5}+1}{2}}$ and $y = \pm\sqrt{\dfrac{\sqrt{5}-1}{2}}$
Yet the third line gives $2xy=2$ so x and y have the same sign.
Finally $(x,y)\in\left\{\left((\sqrt{\dfrac{\sqrt{5}+1}{2}},\sqrt{\dfrac{\sqrt{5}-1}{2}}\right),\left(-\sqrt{\dfrac{\sqrt{5}+1}{2}},-\sqrt{\dfrac{\sqrt{5}-1}{2}}\right)\right\}$
Synthesis
We only have 2 possibilities, therefore since there are only two solutions these are the solutions :

$$\boxed{\delta = \pm\left(\sqrt{\dfrac{\sqrt{5}+1}{2}} + i\sqrt{\dfrac{\sqrt{5}-1}{2}}\right)}$$

3.2.10 Correction of Ex 17

Let $z\in\mathbb{C}$. $\left(z+\dfrac{b}{2a}\right)^2 = \dfrac{b^2-4ac}{4a^2} \Leftrightarrow z^2+\dfrac{b^2}{4a^2}+\dfrac{zb}{a} = \dfrac{b^2}{4a^2}-\dfrac{c}{a} \Leftrightarrow P(z)=0$.
If $\Delta\neq 0$ then $P(z)=0 \Leftrightarrow z+\dfrac{b}{2a} = \pm\dfrac{\delta}{2a} \Leftrightarrow z = \dfrac{-b\pm\delta}{2a}$

4 Linear Algebra

4.1 General Algebra

Groups

A **group** $(G, *)$ is a the pair of a set G on which is defined an operation $*$ that verifies :

- *associativity* : $\forall (a, b, c) \in G^3, \quad a * (b * c) = (a * b) * c$
- *neutral element* : $\exists e \in G, \quad \forall a \in G, \quad a * e = e * a = a$
- *inversibility* : $\forall a \in G, \quad \exists b \in G, \quad a * b = b * a = e$

You can write ab instead of $a * b$ and talk about the group G instead of the group $(G, *)$ when there is no ambiguity.

Ex 29

Let $(G, *)$ be a group, and $(a, b) \in G$. What is $(a * b)^{-1}$?
We now suppose that $\forall a \in G, \quad a^2 = e$. Prove that G is commutative.

Sub-Groups

Let $(G, *)$ be a group, and $H \subset G$. $(H, *)$ is said to ba a sub-group of G when :

$e \in H$ and $\forall (a, b) \in H^2, \quad ab^{-1} \in H$

To prove that H is a sub-group of G you can also prove $H \neq \varnothing$ and $\forall (a, b) \in H^2, \quad ab^{-1} \in H$ and instead of the second proposition you can prove seperately that $\forall a \in H, \quad a^{-1} \in H$ and that $\forall (a, b) \in H^2, \quad a * b \in H$

Morphims

Let $(G, *_G)$ and $(H, *_H)$ be two groups. A map $\varphi : G \longrightarrow H$ is said to be a **group morphism** when :

$\forall (a, b) \in G^2, \quad \varphi(a *_G b) = \varphi(a) *_H \varphi(b)$.

We define its **Kernel** $\mathrm{Ker}(\varphi) = \{a \in G : \varphi(a) = e_H\} = \varphi^{-1}(\{e_H\})$ (where e_H is the neutral element of H).

$\boxed{\text{A morphism } \varphi : G \longrightarrow H \text{ is injective if and only if } \mathrm{Ker}\varphi = \{e_G\}.}$

We also define its **Image** $\mathrm{Im}(\varphi) = \varphi(G)$.

A morphism from a group G to the same groupe G is said to be an **endomorphism**.

A bijective morphism is called an **isomorphism**.

A bijective endomorphism is called an **automorphism**.

Ex 30

Let $(G, *_G)$ and $(H, *_H)$ be two groups and $\varphi : G \longrightarrow H$ be a group morphism. Let G' be a subgroup of G and H' a subgroup of H.

Prove that $\varphi(G')$ is a sub-group of H and that $\varphi^{-1}(H')$ is a sub-group of G.

Rings

A triplet $(A, +, \times)$ is said to be a **ring** when :

- $(A, +)$ is a commutative group : $\forall (a, b) \in A^2, \quad a + b = b + a$
- $\times$ is associative and has a neutral element written 1_A.
- $\times$ is distributive over $+$: Let $(a, b, c) \in A^3$: $a \times (b + c) = (a \times b) + (a \times c)$ and $(a + b) \times c = (a \times c) + (b \times c)$

Let $(A, +_A, \times_A)$ and $(B, +_B, \times_B)$ be two rings. A map $\varphi : A \longrightarrow B$ is said to be a **ring morphism** when :

- φ is a group morphism from the group $(A, +_A)$ to the group $(B, +_B)$
- $\varphi(1_A) = 1_B$
- $\forall (x, y) \in A^2, \quad \varphi(x \times_A y) = \varphi(x) \times_B \varphi(y)$

Definition

An **integral domain** is a ring $(A, +, \times)$ that verifies the properties :

- $A \neq \{0\}$
- $(A, \times)$ is commutative
- A is **integral** : $\forall (x, y) \in A^2, \quad xy = 0 \Leftrightarrow (x = 0 \text{ or } y = 0)$.

By contraposition, the "integral" property can be re-written $\forall (x, y) \in A^2, (x \neq 0 \text{ and } y \neq 0) \Leftrightarrow xy \neq 0$.

Sub-rings

Let $(A, +, \times)$ be a ring. $B \subset A$ is said to be a **sub-ring** of A when :

- $1_A \in B$
- $\forall (x, y) \in B^2, \quad x - y \in B$
- $\forall (x, y) \in B^2, \quad xy \in B$

Therefore a sub-ring is a sub-group that is stable by multiplication (the second operation).

Equivalence relations

A relation $\mathcal{R}$ on a set A is a map $\mathcal{R} : A \longrightarrow \{ \text{ True, False } \}$. An **equivalence relation** verifies :

- *reflexivity* : $\forall a \in A, \quad a\mathcal{R}a$
- *symmetry* : $\forall (a, b) \in A^2, \quad a\mathcal{R}b \Rightarrow b\mathcal{R}a$
- *transitivity* : $\forall (a, b, c) \in A^3, \quad \text{if } a\mathcal{R}b \quad \text{and } b\mathcal{R}c \quad \text{then } a\mathcal{R}c$.

Let $a \in A$. Its equivalency class $\overline{a}$ is defined by $\overline{a} = \{b \in A : a\mathcal{R}b\}$.

Quotient Sets

Let $\mathcal{R}$ be an equivalence relation on a set X. We define the **quotient set** $X/\mathcal{R}$ as the set of the equivalency classes.

For instance for $n \in \mathbb{N}$, $\mathbb{Z}/n\mathbb{Z}$ is defined as the classes of congruence modulo n. It can be given a group structure for $+$ and even a ring structure with $+$ and $\times$.

Fields

A **field** $(\mathbb{K}, +, \times)$ is ring that has the properties :

- $\mathbb{K} \neq \{0\}$
- $(\mathbb{K}, \times)$ is commutative
- $\forall x \in \mathbb{K} \setminus \{0\}, \quad \exists y \in \mathbb{K}, \quad xy = 1_{\mathbb{K}}$: every $x \in \mathbb{K} \setminus \{0\}$ has an inverse for "$\times$".

4.2 Vector Spaces : first definitions

Let $\mathbb{K}$ be field (in practice, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$).

Definition

A **vector space** is a commutative group $(E, +)$ with an external scalar multiplication $\left\{ \begin{array}{ccc} \mathbb{K} \times E & \longrightarrow & E \\ (\lambda, x) & \longmapsto & \lambda.x \end{array} \right.$ that satisfies the following properties :

- *Pseudo-associativity* : $\forall (\lambda, \mu, x) \in \mathbb{K}^2 \times E, \quad \lambda(\mu x) = (\lambda\mu)x$
- *Distributivities* : $\forall (\lambda, \mu, x, y) \in \mathbb{K}^2 \times E^2, \quad \lambda(x + y) = \lambda x + \lambda y,$ and $(\lambda + \mu)x = \lambda x + \mu y$
- *Neutral operator* : $\forall x \in E, \quad 1_{\mathbb{K}} x = x$

The elements of E are the **vectors** and the elements of $\mathbb{K}$ the **scalars**.

Sub-vector spaces

Let E be a $\mathbb{K}$-vectorial space, and let $F \subset E$. F is a **sup-vector space** (or "sub-space") of E when F is stable by addition and by scalar multiplication. This is summarised by :

$$\boxed{0_E \in F, \quad \text{and} \quad \forall (\lambda, \mu, x, y) \in \mathbb{K}^2 \times F^2, \quad \lambda x + \mu y \in F}$$

Since a sub-space is a vector space, to prove that a set is a vector space you can prove that it is a sub-vector space of a larger vector space.

Definition

Let E be a $\mathbb{K}$-vector space, let $n \in \mathbb{N}$ and let $(x_1, ..., x_n) \in E^n$. A **linear combination** of the vectors $x_1, ..., x_n$ is a vector $\sum_{i=1}^{n} \lambda_i x_i$ with $(\lambda_1, ..., \lambda_n) \in \mathbb{K}^n$.

Span

Let $A \subset E$. The **span** of A is the smallest sub-space of E that contains A. It is also the set of all the linear combinations of elements of A. We note it $\text{Span}(A)$.

Linear maps

Let E, F be two $\mathbb{K}$-vector spaces. A **linear map** $f : E \longrightarrow F$ is a map that is :

- compatible with "+" : $\forall(x,y) \in E^2, \quad f(x+y) = f(x) + f(y)$ (so f is a group morphism)
- compatible with "." : $\forall(\lambda, x) \in \mathbb{K} \times E, \quad f(\lambda x) = \lambda f(x)$

f is a linear map if and only if $\boxed{\forall(\lambda,\mu,x,y) \in \mathbb{K}^2 \times E^2, \quad f(\lambda x + \mu y) = \lambda f(x) + \mu f(y)}$

The image of any sub-vector space by a linear map and its kernel are sub-vector spaces.

A linear map f is a group morphism therefore is $\boxed{\text{injective if and only if } \mathrm{Ker} f = \{0\}}$

When a linear map f goes from E to E it is called an **endomorphism**.
When it is bijective we call it an **isomorphism**.
When it is both bijective and an endomorphism we call it an **automorphism**.

Definition

We note $L(E)$ the set of all the endomorphisms of E (it's a $\mathbb{K}$-vector space).
We note $L(E, F)$ the vector space of the linear maps from E to F.
We note $GL(E)$ the set of all the automorphisms of E (it's a group for the composition "$\circ$" but not a vector space).

Exercises for next time : We note $O_{\mathbb{R},\mathbb{R}}$ the set of the odd functions from $\mathbb{R}$ to $\mathbb{R}$.

- Prove that $(O_{\mathbb{R},\mathbb{R}}, +)$ is a sub-group of $(F(\mathbb{R},\mathbb{R}), +)$.
- Prove that $(O_{\mathbb{R},\mathbb{R}}, \circ)$ is a sub-group of $(F(\mathbb{R},\mathbb{R}), \circ)$.
- Is $(O_{\mathbb{R},\mathbb{R}}, \times)$ a sub-group of $(F(\mathbb{R},\mathbb{R}), \times)$?
- Prove that $(O_{\mathbb{R},\mathbb{R}}, +, \circ)$ is a sub-ring of $(F(\mathbb{R},\mathbb{R}), +, \circ)$.
- Prove that $(O_{\mathbb{R},\mathbb{R}}, +, .)$ is a $\mathbb{R}$-sub-vector space of $F(\mathbb{R},\mathbb{R})$.
- Prove that $\mathbb{Z}/3\mathbb{Z}$ is a field.
- Prove that $z \longmapsto \overline{z}$ is an automorphism of the $\mathbb{R}$-vector space $\mathbb{C}$ and of the ring $(\mathbb{C}, +\times)$.
- Prove that matrix transposition is an injective linear map.

5 Vector Spaces

5.1 Vector Families

We consider a family $(e_i)_{i \in [\![1,n]\!]}$ of vectors of a $\mathbb{K}$-vector space E.

Definition

We say that (e_i) is a **spanning** family when $\mathrm{Span}((e_i)) = E$. This is the same as :

All vectors are linear combinations of the (e_i) : $\boxed{\forall x \in E, \quad \exists(\lambda_1, ..., \lambda_n) \in \mathbb{K}^n \quad x = \sum_{i=1}^{n} \lambda_i e_i}$

We say that (e_i) is spanning a sub-vector space F when $\mathrm{Span}((e_i)) = F$.

independent families

(e_i) is said to be **linearly independent** when all vectors can have only one decomposition in that family :

$$\forall x \in E, \quad x = \sum_{i=1}^{n} \lambda_i e_i = \sum_{i=1}^{n} \mu_i e_i, \quad \Rightarrow \quad \forall i \in [\![1, n]\!], \quad \lambda_i = \mu_i$$

Equivalently, $\boxed{\forall (\lambda_1, ..., \lambda_n) \in \mathbb{K}^n, \quad \sum_{i=1}^{n} \lambda_i e_i = 0 \quad \Rightarrow \quad \forall i \in [\![1, n]\!], \quad \lambda_i = 0}$

A family that isn't independent is said to be **dependent**.

Two vectors are independent if and only if they are not colinear.

Definition

(e_i) is a **basis** of E when it is both spanning E and independent.

This is summarised by : $\forall x \in E, \quad \exists! (\lambda_1, ..., \lambda_n) \in \mathbb{K}^n, \quad x = \sum_{i=1}^{n} \lambda_i e_i$

This is generalised to infinite families by replacing each sum by one containing only a finite number of (e_i) : for example $(e_i)_{i \in E}$ is a basis of E (not finite-dimensional here) when :

$$\forall x \in E, \quad \exists p \in \mathbb{N}, \quad \exists (\lambda_{a_1}, ..., \lambda_{a_p}) \in \mathbb{K}^p, \quad x = \sum_{i=1}^{p} \lambda_{a_i} e_{a_i}.$$

Characterisation of a linear map by the image of a basis

Let E, F be two $\mathbb{K}$-vector spaces and $(e_j)_{j \in [\![1,p]\!]}$ be a basis of E and $(y_i)_{i \in [\![1,p]\!]}$ a family of vectors of F.

Then $\boxed{\exists! f \in L(E, F), \quad \forall j \in [\![1, p]\!], \quad f(e_j) = y_j}$

Image of a basis by a linear map

Let E, F be two $\mathbb{K}$-vector spaces and $(e_i)_{i \in [\![1,n]\!]}$ a basis of E. Let $f \in L(E, F)$. We have :

$$f \text{ is injective} \Leftrightarrow (f(e_i)) \text{ is independent}$$

$$f \text{ is surjective} \Leftrightarrow (f(e_i)) \text{ is spanning}$$

$$f \text{ is an isomorphism} \Leftrightarrow (f(e_i)) \text{ is a basis of } F$$

5.2 Sums of Sub-spaces

Let E be a $\mathbb{K}$-vector space and E_1, E_2 be two sub-vector spaces of E.

We define $\varphi : \begin{cases} E_1 \times E_2 & \longrightarrow & E \\ (x_1, x_2) & \longmapsto & x_1 + x_2 \end{cases}$ a linear map from $E_1 \times E_2$ to E.

Direct Sum

The set $E_1 + E_2 = \{x_1 + x_2 | (x_1, x_2) \in E_1 \times E_2\}$ is a sub-space of E (because $E_1 + E_2 = \text{Im}\varphi$)

Let $F = E_1 + E_2$. Because they both equate to "φ is injective", we have *1)* $\Leftrightarrow$ *2)* :

1) $\forall x \in F, \quad x = x_1 + x_2 = x_1' + x_2' \Rightarrow x_1 = x_1', x_2 = x_2' \quad (x_1, x_1' \in E_1, x_2, x_2' \in E_2)$

2) $\forall x \in F, \quad \exists!(x_1, x_2) \in E_1 \times E_2, \quad x = x_1 + x_2$

In that case we say that E_1 and E_2 are in **direct sum** and we write $E_1 \oplus E_2 = F$.

We have $\boxed{\text{"}E_1 \text{ and } E_2 \text{ are in direct sum"} \Leftrightarrow E_1 \cap E_2 = \{0\}}$

When $E_1 \oplus E_2 = E$ we say that they are **supplementary**.

Definition on a decomposition

Let $E_1, ..., E_p$ be subspaces of E so that $E = \bigoplus_{i=1}^{p} E_i$ and $(f_1, ..., f_p) \in L(E_1, F) \times ... \times L(E_p, F)$.

Then $\exists! f \in L(E, F), \quad \forall i \in [\![1, p]\!], \quad f|_{E_i} = f_i$

projectors

Suppose $E = F \oplus G$. Since $\forall x \in E, \quad \exists!(x_F, x_G) \in F \times G, \quad x = x_F + f_G$, we can define the **projector** on F parallelly to G :

$$p : \left\{ \begin{array}{ccc} E & \longrightarrow & E \\ x & \longmapsto & x_F \end{array} \right.$$

p is an endomorphism and we have $\text{Ker}(p) = G$, $\text{Im}(p) = F$ and $p^2 = p$. ($p^2 = p \circ p$)

Definition

A **linear form** is a linear map $E \longrightarrow \mathbb{K}$.

Hyperplanes

A subspace H of E is said to be a **hyperplane** of E when one of the following equivalent properties is met :

1) $\exists N \in E \setminus \{0\}, \quad E = H \oplus \mathbb{K}N$ ($\mathbb{K}N = \{\lambda N | \lambda \in \mathbb{K}\}$)

2) $\exists \varphi \in L(E, \mathbb{K}) \setminus \{0\}, \quad H = \text{Ker}\varphi$

Adapted basis theorem

Suppose $E = \bigoplus_{k=1}^{p} E_k$ and that it has a basis. Then there exists an **adapted** basis $(b_i)_{i \in I}$:

$I = \bigcup_{k=1}^{p} I_k$ with the I_k disjoint,

and $\forall k \in [\![1, p]\!], \quad (b_i)_{i \in I_k}$ is a basis of E_k.

5.3 Finite Dimensional Vector Spaces

Definition

A vector space E is said to be **finite-dimensional** when it has a finite spanning family.

Let E be a finite-dimensional $\mathbb{K}$-vector space.

Completion theorems

- Any independent family of vectors of E can be completed into a basis of E. That completion can be made with vectors of any spanning family.
- Any family spanning E contains a basis of E.
- E has a basis.

The oversize lemma

Suppose that E is spanned by a family of n vectors. Then all families of $n+1$ vectors are dependent.

Dimension

All bases of E have the same size n, we call it the **dimension** of E ($\dim E = n$).

All independent families of size p satisfy $p \leq n$ with equality if and only if it is a basis

All spanning families of size p satisfy $p \geq n$ with equality if and only if it is a basis

Dimension of sub-spaces

Let F be a sub-space of E. Then F is finite-dimensional. Let $p = \dim F$, we have $p \leq n$ with equality if and only if $E = F$.

Dimension of Sums

Let F, G be two subspaces of E. We have :

- $\dim(F+G) = \dim F + \dim G - \dim(F \cap G)$ (Grassmann's formula)
- if F and G form a direct sum then $\dim(F \oplus G) = \dim F + \dim G$
- $\dim(F+G) = \dim F + \dim G \Leftrightarrow F$ and G are in direct sum.

Rank

We define the **rank** of a family (x_i) by $\text{rank}(x_i) = \dim(\text{Span}(x_i))$.

We define the **rank** of a linear map f by $\text{rank}(f) = \dim(\text{Im} f)$

Let $f \in L(E, F)$ where E is of finite dimension n and F is any vector space.
We have the **rank theorem** : $\boxed{\text{rank}(f) + \dim(\text{Ker}(f)) = \dim E}$

Let $f \in L(E)$. f is injective $\Leftrightarrow$ f is surjective $\Leftrightarrow$ f is an isomorphism $\Leftrightarrow$ $\text{rank}(f) = n$.

5.4 Exercises

Ex 31

Let E and F be two vector spaces.

Find all the linear maps $u : E \longrightarrow F$ so that $\forall x \in E, \quad (u(x), x)$ is dependent.

Ex 32

Let E and F be two finite-dimensional vector spaces.

Let $u \in L(E, F)$ and $A \subset E$. Prove that $\text{Span}(u(A)) = u(\text{Span}A)$.

Ex 33

Let E be a finite-dimension vector space of dimension n.

Let $(f, g) \in L(E)^2$ so that $f + g = \text{Id}_E$ and $\text{rank}(f) + \text{rank}(g) \leq n$ (Id_E is the identity map $x \longmapsto x$)

1) Let $F = \text{Im}(f)$ and $G = \text{Im}(g)$. Prove that $F \oplus G = E$.

2) Using the rank theorem prove that $\text{Ker}(f) + \text{Ker}(g) = E$.

3) Prove that f and g are projectors.

Ex 34

Let E be a vector space and $f \in L(E)$. Let $(\alpha, \beta) \in \mathbb{K}^2$ so that $\alpha \neq \beta$.

1) Prove that $E = \text{Im}(f - \alpha\text{Id}_E) + \text{Im}(f - \beta\text{Id}_E)$.

2) We now suppose that $(f - \alpha\text{Id}_E) \circ (f - \beta\text{Id}_E) = 0$. Prove that f is an automorphism and give f^{-1}.

3) Prove that $E = \text{Ker}(f - \alpha\text{Id}_E) \oplus \text{Ker}(f - \beta\text{Id}_E)$.

6 Matrices

6.1 Matrix - Linear map correspondance

$M_{n,p}(\mathbb{K})$

$M_{n,p}(\mathbb{K})$ is the set of all the matrices with n lines and p columns with coefficients in $\mathbb{K}$. It is a $\mathbb{K}$-vector space with two multiplications :

$$\left\{ \begin{array}{ccc} M_{n,p}(\mathbb{K}) \times M_{p,q}(\mathbb{K}) & \longrightarrow & M_{n,q}(\mathbb{K}) \\ (A, B) & \longmapsto & A \times B \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{ccc} M_{n,p}(\mathbb{K}) \times \mathbb{K}^p & \longrightarrow & \mathbb{K}^n \\ (A, X) & \longmapsto & A \times X \end{array} \right.$$

Let E and F be two finite-dimensional vector spaces of dimensions p and n respectively. Let $(e_j)_{j \in [\![1,p]\!]}$ be a basis of E and $(f_i)_{i \in [\![1,n]\!]}$ a basis of F.

Correspondance

Let $x = \sum_{i=1}^{n} x_i f_i \in F$. We define $\boxed{\text{mat}_{(f_i)}(x) = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}}$ It is an element of $\mathbb{K}^n$ or of $M_{n,1}(\mathbb{K})$

Let $(y_1, ..., y_p) \in F^p$. We define $\boxed{\text{mat}_{(f_i)}(y_1, ..., y_p) = \left(\begin{array}{c|c|c} Y_1 & ... & Y_p \end{array} \right)}$

Where each Y_j is the column matrix of y_j in the basis (f_i). $\text{mat}_{(f_i)}(y_j)$ is an element of $M_{n,p}(\mathbb{K})$.

Let $u \in L(E, F)$. We define $\boxed{\text{mat}_{(e_j),(f_i)}(u) = \left(\begin{array}{c|c|c} u(e_1) & ... & u(e_p) \end{array} \right) \in M_{n,p}(\mathbb{K})}$

We have $\text{mat}_{(e_j),(f_i)}(u) = (a_{i,j})_{i \in [\![1,n]\!], j \in \{1,p\}}$ with $\forall j \in \{1, p\}, \quad u(e_j) = \sum_{i=1}^{n} a_{i,j} f_i$

Let $u \in L(E, F)$. Let $A = \text{mat}_{(e_j),(f_i)}(u)$. Let $x \in E$ and $y \in F$. Let $X = \text{mat}_{(e_j)}(x)$ and $Y = \text{mat}_{(f_i)}(y)$.

We have $\boxed{AX = Y \Leftrightarrow u(x) = y}$

Canonical association

Consider (e_j) the canonical basis of $\mathbb{K}^p$ and (f_i) the canonical basis of $\mathbb{K}^n$.

Let $A \in M_{n,p}(\mathbb{K})$. Its **canonically associated endomorphism** is $u : \begin{cases} \mathbb{K}^p & \longrightarrow & \mathbb{K}^n \\ X & \longrightarrow & AX \end{cases}$

We therefore define $\text{Im}(A) = \text{Im}(u)$ and $\text{Ker}(A) = \text{Ker}(u)$.

Composition and matrix products

Let E, F, G be finite-dimensional spaces of bases B, C, D. Let $u \in L(E, F)$ and $v \in L(F, G)$.
We have : $\boxed{\text{mat}_{B,D}(v \circ u) = \text{mat}_{C,D}(v) \times \text{mat}_{B,C}(u)}$

Block matrices and stability

Let F be a subset of E and $u \in L(E)$. We say that F is **stable by** u when $u(F) \subset F$.
Suppose $E = E_1 \oplus E_2$. Let (b_i) be an adapted basis : $(b_1, ..., b_k)$ is a basis of E_1 and $(b_{k+1}, ..., b_p)$ of E_2.

$\text{mat}_{(b_i)}(u)$ can be written in block form : $\text{mat}_{(b_i)}(u) = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right)$

Then $\boxed{E_1 \text{ is stable by } u \Leftrightarrow C = 0}$ and $\boxed{E_2 \text{ is stable by } u \Leftrightarrow B = 0}$

6.2 Specific Matrices

$M_{n,p}(\mathbb{K})$ has a canonical basis : $(E_{i,j})_{i \in [\![1,n]\!], j \in [\![1,p]\!]}$ so that $(E_{i,j})_{k,l} = \delta_{i,k}\delta_{j,l}$.
Therefore $\dim (M_{n,p}(\mathbb{K})) = np$
We now consider $n = p$: the **square matrices**.

Definition

We define $I_n = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$

We define $GL_n(\mathbb{K})$ the group of the inversible matrices :
$\forall A \in GL_n(\mathbb{K}), \quad \exists B \in GL_n(\mathbb{K}), \quad AB = BA = I_n$

We define the **trace** Tr : $\left\{ \begin{array}{ccc} M_n(\mathbb{K}) & \longrightarrow & \mathbb{K} \\ A & \longmapsto & \sum_{i=1}^{n} a_{i,i} \end{array} \right.$

We also define the matrix transposition $\left\{ \begin{array}{ccc} M_n(\mathbb{K}) & \longrightarrow & M_n(\mathbb{K}) \\ A & \longmapsto & A^T = (a_{j,i})_{(i,j)\in[\![1,n]\!]^2} \end{array} \right.$

Properties

Tr is a linear form and $A \mapsto A^T$ is an automorphism. Let $(A, B) \in M_n(\mathbb{K})^2$. We have :

- $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$
- $(AB)^T = B^T A^T$
- $\mathrm{Tr}(A^T) = \mathrm{Tr}(A)$

Matrix types

A matrix $(a_{i,j})$ is said to be **diagonal** when $\forall (i,j) \in [\![1,n]\!]^2, \quad i \neq j \Rightarrow a_{i,j} = 0$. We note the vector space of the diagonal matrices of size n $D_n(\mathbb{K})$.

A matrix $(a_{i,j})$ is said to be **upper-triangular** when $\forall (i,j) \in [\![1,n]\!]^2, \quad i > j \Rightarrow a_{i,j} = 0$. It is said to be **strictly upper-triangular** when is it both upper-triangular and when its diagonal is 0.
The vector space of the upper-triangular matrices is written $T_n^+(\mathbb{K})$ and the strictly upper-triangular ones $T_n^{++}(\mathbb{K})$.
Similarly we define the **lower-triangular** matrices $T_n^-(\mathbb{K})$ and their strict versions $T_n^{--}(\mathbb{K})$.

A matrix A is said to be **symmetrical** when $A^T = A$: their vector space is written $S_n(\mathbb{K})$.
A matrix A is said to be **antisymmetrical** when $A^T = -A$: their vector space is written $A_n(\mathbb{K})$.

Remark : Let $(A, B) \in T_n^+(\mathbb{K})^2$. $AB \in T_n^+(\mathbb{K})$ with $\forall i \in [\![1,n]\!], \quad (AB)_{i,i} = a_{i,i} b_{i,i}$

Endomorphism-matrix relation

Let E be finite-dimensional of dimension n and of basis $(e_i)_{i\in[\![1,n]\!]}$. Let $(x_i) \in E^n$ and let $A = \mathrm{mat}_{(e_i)}(x_i)$.

$\boxed{(x_i) \text{ is a basis of } E \Leftrightarrow A \text{ is inversible}}$

Then let E, F two finite-dimensional spaces of bases (e_j) and (f_i), let $u \in L(E, F)$.

$\boxed{\mathrm{mat}_{(e_j),(f_i)}(u) \text{ is inversible} \Leftrightarrow u \text{ is an isomorphism}}$

Therefore for square matrices if $AB = I_n$ then A is inversible of inverse B.

Inversibility of a triangular matrix

Let $T \in T_n^+(\mathbb{K})$. $\boxed{T \text{ is inversible} \Leftrightarrow \text{its diagonal has no zero}}$

In that case, $T^{-1} \in T_n^+(\mathbb{K})$.

Nilpotence of a strictly triangular matrix

We say that a matrix A is **nilpotent** when $\exists k \in \mathbb{N}, \quad A^k = 0$. (Same for an endomorphism).

Let $T \in T_n^{++}(\mathbb{K})$. Then $T^n = 0$.

6.3 Rank, equivalence, similarity

Changing bases

Let E be finite-dimensional with bases (e_i) and (e'_i). The base-change matrix from (e_i) to (e'_i) is $\boxed{P_{(e'_i)\leftarrow(e_i)} = \mathrm{mat}_{(e_i)}(e'_j)}$

Let $x \in E$, let $X = \mathrm{mat}_{(e_i)}(x)$ and $X' = \mathrm{mat}_{(e'_i)}(x)$. We have $\boxed{X = P_{(e'_i)\leftarrow(e_i)}X'}$

Let $u \in L(E, F)$ with (e_j) and (e'_j) two bases of E, (f_i) and (f'_i) two bases of F.
Let $A = \mathrm{mat}_{(e_j),(f_i)}(u)$ and $A' = \mathrm{mat}_{(e'_j),(f'_i)}(u)$

Then $\boxed{A' = \left(P_{(f'_i)\leftarrow(f_i)}\right)^{-1} \times A \times P_{(e'_j)\leftarrow(e_j)}}$

Definition

Two matrices A and B in $M_{n,p}(\mathbb{K})$ are said to be **equivalent** when :
$\boxed{\exists (P, Q) \in GL_n(\mathbb{K}) \times GL_p(\mathbb{K}), \quad B = PAQ}$ This means that they represent the same linear map in the right pair of bases.

Two square matrices A and B are said to be **similar** when $\boxed{\exists P \in GL_n(\mathbb{K}), \quad A = P^{-1}BP}$.
This means that they represent the same endomorphism in the right basis.

They are both equivalence relations.

Example : two similar matrices have the same trace.

Rank of a matrix

Let $A \in M_{n,p}(\mathbb{K})$. We define $\mathrm{rank}(A) = \mathrm{rank}(u)$ (with $u \in L(\mathbb{K}^p, \mathbb{K}^n)$ its canonically associated linear map)

Let $r \leq \min(n, p)$. We define by blocks $J_r = \left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right)$

Let $A \in M_{n,p}(\mathbb{K})$. $\boxed{\mathrm{rank} A = r \Leftrightarrow A \text{ is equivalent to } J_r}$

Example : $\mathrm{rank}(A) = \mathrm{rank}(A^T)$.

Rank and subs-matrices

Let $A \in Mn, p(\mathbb{K})$.

- Suppose $\operatorname{rank}(A) \geq r$. Then A has an inversible sub-matrix of size $r \times r$.
- Suppose that A has an inversible sub-matrix of size $r \times r$. Then $\operatorname{rank}(A) \geq r$.

Sub-matrix theorem

$\operatorname{rank}(A) = r \Leftrightarrow$ "A has an inversible sub-matrix of size (r, r) and all strictly bigger square sub-matrices are not inversible."

6.4 TD

Ex 35

Let $M = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 3 \end{pmatrix}$, $B = (e_1, e_2, e_3)$ and the canonical basis of $\mathbb{R}^3$

Let $\begin{cases} u_1 & = & e_1 \\ u_2 & = & e_2 + e_2 + e_3 \\ u_3 & = & e_3 \end{cases}$, and $U = (u_1, u_2, u_3)$.

1) Show that U is a basis of $\mathbb{R}^3$.

2) Give $P = P_{U \leftarrow B}$ the change of basis matrix from B to U and compute P^{-1}.

3) Let $f \in L(\mathbb{R}^3)$ canonically associated to M. Give $N = \operatorname{mat}_U(f)$.

4) Compute N^n and show $M^n = PN^nP^{-1}$.

Ex 36

Let $(A, B) \in M_n(\mathbb{C})^2$, let $a = \operatorname{Tr}(A)$ and $b = \operatorname{Tr}(b)$.
Let (E) be the equation $X = \operatorname{Tr}(X)A + B, \quad X \in M_n(\mathbb{C})$.

1) We consider that $a \neq 1$, solve (E).

2) We consider that $a = 1$, prove that if $b \neq 0$, (E) has no solution.

3) We consider $(a, b) = (1, 0)$ until the end.
Prove that $\forall X \in M_n(\mathbb{C}), X = \operatorname{Tr}(X)A + B \Leftrightarrow Y = \operatorname{Tr}(Y)A$ with $Y = X - B$.
We call the second equation (F) (of variable $Y \in M_n(\mathbb{C})$)

4) Solve (F) and then solve (E).

Ex 37

HADAMARD's Lemma :

Let $n \in \mathbb{N}^*$ and $A \in M_n(\mathbb{C})$ so that $\forall i \in [\![1, n]\!], \quad |a_{i,i}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{i,j}|$

Prove that A is inversible.

Hint : study $\operatorname{Ker}(A)$

Ex 38

Let $n \geq 2$, $A \in M_n(\mathbb{K})$ and $B = \mathrm{Adj}(A)$.

1) We suppose $\mathrm{rank}(A) = n$. Find $\mathrm{rank}(B)$.

2) We suppose $\mathrm{rank}(A) = n-1$. Find $\mathrm{rank}(B)$.

3) We suppode $\mathrm{rank}(A) < n-1$. Find $\mathrm{rank}(B)$.

Ex 39

Let $A \in M_n(\mathbb{K})$. Prove that $\mathrm{rank}(A) = 1 \Leftrightarrow \exists C \in \mathbb{K}^n, \quad \exists L \in M_{1,n}(\mathbb{K}), \quad A = CL$

7 Euclidian Spaces

7.1 Inner products

Let E be an $\mathbb{R}$-vector space

Definition

An **inner product** $(\cdot,\cdot)$ is an map $E^2 \longrightarrow \mathbb{R}$ that satisfies :

- *symmetry* : $\forall (x,y) \in E^2, \quad (x|y) = (y|x)$
- *bilinearity* : $\forall (x,y,z) \in E^3, \quad \forall (\lambda,\mu) \in \mathbb{R}^2, \quad (\lambda x + \mu y|z) = \lambda(x|z) + \mu(y|z)$
- *positivity* : $\forall x \in E, \quad (x|x) \geq 0$
- *definiteness* : $\forall x \in E, \quad (x|x) = 0 \Rightarrow x = 0$

Examples :

$$\left\{\begin{array}{ccc} (\mathbb{R}^n)^2 & \longrightarrow & \mathbb{R} \\ (X,Y) & \longmapsto & \sum_{i=1}^n x_i y_i \end{array}\right. \qquad \left\{\begin{array}{ccc} M_n(\mathbb{R})^2 & \longrightarrow & \mathbb{R} \\ (A,B) & \longmapsto & \mathrm{Tr}(A^T B) \end{array}\right. \qquad \left\{\begin{array}{ccc} (C^0([0,1],\mathbb{R}))^2 & \longrightarrow & \mathbb{R} \\ (f,g) & \longmapsto & \int_0^1 f(t)g(t)\mathrm{d}t \end{array}\right.$$

Definition

A **real pre-hilbertian** space is a $\mathbb{R}$-vector space with an inner product $(\cdot,\cdot)$.

We define the **euclidian norm** on E : $\left\{\begin{array}{ccc} E & \longrightarrow & \mathbb{R}_+ \\ x & \longmapsto & \|x\| = \sqrt{(x|x)} \end{array}\right.$

We define the **euclidian distance** between two vectors x, y : $\|x-y\|$

We now consider E to be a real pre-hilbertian space.

Cauchy-Schwarz inequality

$\boxed{\forall (x,y) \in E^2, \quad |(x|y)| \leq \|x\|\|y\|}$ with equality if and only if x and y are colinear.

Examples

$$\left|\sum_{i=1}^n x_i y_i\right| \leq \sqrt{\left(\sum_{i=1}^n x_i^2\right)\left(\sum_{i=1}^n y_i^2\right)}, \qquad \left|\int_a^b f(t)g(t)\mathrm{d}t\right| \leq \sqrt{\left(\int_a^b f(t)^2\mathrm{d}t\right)\left(\int_a^b g(t)^2\mathrm{d}t\right)}$$

Properties of the inner product and of the euclidian norm

Let $(x, y) \in E^2$.

- **Triangular Inequality** : $\|x + y\| \leq \|x\| + \|y\|$
  Polarisation identities :
- $2(x|y) = \|x + y\|^2 - \|x\|^2 - \|y\|^2$
- $2(x|y) = \|x\|^2 + \|y\|^2 - \|x - y\|^2$
- $4(x|y) = \|x + y\|^2 - \|x - y\|^2$
- **Parallelogram identity** : $\|x + y\|^2 + \|x - y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right)$

7.2 Orthogonality

Definition

Two vectors x, y are **orthogonal** when $(x|y) = 0$. We write $x \perp y$

Two subsets A, B of E are **orthogonal** when $\forall (a, b) \in A \times B, \quad (a|b) = 0$. We write $A \perp B$

Let A be a subset of E. Its **orthogonal** $A^\perp$ is the set of all the vectors that are orthogonal to A.

Let $A \subset E$. We have :

- $A \perp A^\perp$
- $B \subset A^\perp \Leftrightarrow A \perp B \Leftrightarrow A \subset B^\perp$
- $A^\perp$ is a subspace of E
- $A \subset B \Rightarrow B^\perp \subset A^\perp$
- $\mathrm{Span}(A)^\perp = A^\perp$
- $x \in A \cap A^\perp \Rightarrow x = 0$

If F is a finite-dimensional subspace of E then $F \overset{\perp}{\oplus} F^\perp = E$ and $(F^\perp)^\perp = F$.

Ex 40 Prove the six listed properties.

Orthogonal families, othonormal families, orthonormal bases

Let $(e_i)_{i \in [\![1,n]\!]}$ be a family of vectors of E.

(e_i) is said to be **orthogonal** when $\forall i \neq j \in [\![1, n]\!], \quad (e_i|e_j) = 0$

(e_i) is said to be **orthonormal** when it is orthogonal and $\forall i \in [\![1, n]\!], \quad \|x_i\| = 1$

$\boxed{\text{All orthogonal families with nonzero vectors are independent}}$

If (e_i) is an orthonormal basis of E then $\forall x \in E, \quad x = \sum_{i=1}^{n} (x|e_i)e_i$ and $\|x\|^2 = \sum_{1=1}^{n} x_i^2$

Orthogonal Projectors

Let F be a finite-dimensional sub-vector space of E. The **orthogonal projector** on F is the projector on F parallely to $F^\perp$.

Schmidt orthonormalisation

Let $(e_1, ..., e_n)$ be an independent family of E.

There exists an orthonormal family $(f_1, ..., f_n)$ so that $\forall p \in [\![1, n]\!], \quad \text{Span}(e_1, ..., e_p) = \text{Span}(f_1, ..., f_p)$.

7.3 Isometries and orthogonal matrices

We now consider a **euclidian space** (real pre-hilbertian space of finite dimension) E and $n = \dim E$.

Definition

An endomorphism $u \in L(E)$ is said to be an **isometry** when $\forall x \in E, \quad \|u(x)\| = \|x\|$.

Another definition is $\forall (x, y) \in E^2, \quad (u(x)|u(y)) = (x|y)$.

An isometry is also called an **orthogonal automorphism** because it is bijective.

The isometries form a group for $\circ$ called $O(E)$.

Definition

A matrix $O \in M_n(\mathbb{R})$ is said to be **orthogonal** when $O^T O = I_n$.

The orthogonal matrices form a group for $\times$ noted $O_n(\mathbb{R})$.

A matrix $O \in M_n(\mathbb{R})$ is orthogonal if and only if its columns form an orthonormal basis of $\mathbb{R}^n$.

Matrix representation of orthogonality

Let B be an orthonormal basis of E and $u \in L(E)$. The 3 following properties are equivalent :
1) u is an isometry
2) $u(B)$ is an orthonormal basis
3) $\text{mat}_B(u)$ is an orthogonal matrix

Isometries of $\mathbb{R}^2$

Let $u \in O(\mathbb{R}^2)$.

If $\det u = 1$ then it is called a **rotation** and its matrix is in the form $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ in a certain orthonormal basis.

If $\det u = -1$ then it is called a **symmetry** and its matrix is in the form $S_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$ in a certain orthonormal basis.

8 Polynomials

Definition

A **polynomial** on a field $\mathbb{K}$ is a sequence with values in $\mathbb{K}$ and with a finite number of nonzero values.

We note their set $\mathbb{K}[X]$. It is a $\mathbb{K}$-vector space with the following laws :

- If $P = (a_k)_{k\in\mathbb{N}}$ and $Q = (b_k)_{k\in\mathbb{N}}$, $P + Q = (a_k + b_k)_{k\in\mathbb{N}}$
- If $P = (a_k)_{k\in\mathbb{N}}$ and $\lambda \in \mathbb{K}$, $\lambda P = (\lambda a_k)_{k\in\mathbb{N}}$

$\mathbb{K}[X]$ has an internal multiplication : If $P = (a_k)_{k\in\mathbb{N}}$ and $Q = (b_k)_{k\in\mathbb{N}}$, write $C = PQ = (c_k)_{k\in\mathbb{N}}$, we have $\forall k \in \mathbb{N}, \quad c_k = \sum_{i=0}^{k} a_i b_{k-i} = \sum_{i+j=k} a_i b_j$

We define $X = (0, 1, 0, ...) = (\delta_{k,1})_{k\in\mathbb{N}}$. We have $\forall n \in \mathbb{N}, \quad X^n = (\delta_{k,n})_{k\in\mathbb{N}}$.

$\mathbb{K}[X]$ has a **canonical basis** $(X^n)_{n\in\mathbb{N}}$ and all $P \in \mathbb{K}[X]$ can be written $P = \sum_{k=0}^{d} a_k X_k$.

You can also write $P = \sum_{k=0}^{+\infty} a_k X^k$ because the a_k are 0 after a certain rank.

Definition

Let $P = (a_k)_{k\in\mathbb{N}} \neq 0$. We define its **degree** $\boxed{\deg(P) = \max\{k \in \mathbb{N} | a_k \neq 0\}}$

We define $\mathbb{K}_n[X]$ as the vector space of the polynomials of degree smaller than n.

We define $\deg(0) = -\infty$

If P is of degree $n \in \mathbb{N}$ then we can write $P = \sum_{k=0}^{n} a_k X_k$ with $a_n \neq 0$ (called the **dominant coefficient** of P).

Properties of the degree

Let $(P, Q) \in \mathbb{K}[X]^2$. We have :

- $\deg(P + Q) \leq \max(\deg P, \deg Q)$
- if $\deg P \neq \deg Q$ then $\deg(P + Q) = \max(\deg P, \deg Q)$
- $\deg(PQ) = \deg P + \deg Q$

As a consequence of the third point, $\mathbb{K}[X]$ is an integral domain.

Composition

Let $P = \sum_{k=0}^{n} a_k X^k$ and $Q \in \mathbb{K}[X]$. We define $P \circ Q = \sum_{k=0}^{n} a_k Q_k$

If Q is not constant then $\deg(P \circ Q) = \deg P \times \deg Q$

Definition

We say that A **divides** B (we write $A|B$) when $B = AQ$.

A **unitary** polynomial is so that its dominant coefficient is 1.

Two polynomials that are multiples of each other are said to be **associated**.

8.1 Polynomial derivation

Definition

Let $P = \sum_{k=0}^{n} a_k X_k$. We define $P' = \sum_{k=1}^{n} a_k X^{k-1}$.

We have the immediate properties for $(P, Q) \in \mathbb{K}[X]^2$ and $(\lambda, \mu) \in \mathbb{K}^2$:

$(\lambda P + \mu Q)' = \lambda P' + \mu Q'$, $(PQ)' = P'Q + QP'$, $(P \circ Q)' = Q' \times P' \circ Q$

Leibniz's formula

For $(P, Q) \in \mathbb{K}[X]^2$ and $n \in \mathbb{N}$: $\boxed{(PQ)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} P^{(k)} Q^{(n-k)}}$

8.2 Arithmetic in $\mathbb{K}[X]$

Euclidian division in $\mathbb{K}[X]$

Let $(A, B) \in \mathbb{K}[X] \times (\mathbb{K}[X] \setminus \{0\})$. Then $\exists!(Q, R) \in \mathbb{K}[X]^2, \quad A = BQ + R$ with $\deg R < \deg B$

GCDs

Let $(P, Q) \in \mathbb{K}[X]^2$.

A GCD (greatest common dividor) of P and Q is a polynomial D so that $\deg D = \max\{\deg A : A|P \text{ and } A|Q\}$. and $D|P$ and $D|Q$. We write $P \wedge Q$ the only GCD of P and Q that is unitary.

We have $\boxed{A\mathbb{K}[X] + B\mathbb{K}[X] = (A \wedge B)\mathbb{K}[X]}$

LCDs

A LCD (least common denominator) of P and Q is a polynomial $M \neq 0$ so that $\deg(M) = \min\{\deg A : A \neq 0, P|A, Q|A\}$ and $P|M$, $Q|M$. We write $P \vee Q$ the only LCD of P and Q that is unitary.

We have $\boxed{A\mathbb{K}[X] \cap B\mathbb{K}[X] = (A \vee B)\mathbb{K}[X]}$

For example, let $(P, Q) \in \mathbb{K}[X]^2$ and $D = P \wedge Q$. Then $\exists(U, V) \in \mathbb{K}[X]^2, \quad PU + QV = D$ (Bezout's Theorem).

Definition

A polynomial is said to be **irreductible** when its only dividors are constant.

Example : all polynomials of degree 1 or less are irreductible. $X^2 + 1$ is irreductible in $\mathbb{R}[X]$.

Decomposition into Irreductible Factors (DIF)

All nonzero polynomials P can be written $P = \lambda P_1^{\alpha_1} \times ... \times P_n^{\alpha_n}$ with the P_i unitary irreductible polynomials.

8.3 Roots

Definition

We have $P(\lambda) = 0 \Leftrightarrow (X - \lambda)|P$. By extension :

A scalar $\lambda \in \mathbb{K}$ is said to be a **root of order** p when $(X-\lambda)^p|P$ (so $P = (X-\lambda)^p \times Q$)), and p is the maximum number that satisfies this.

A polynomial P of degree n is said to be **totally separated** when it has n roots (counted with order of multiplicity), so $P = a\prod_{i=1}^{n}(X - \lambda_i)$

Characterisations of the root order

$\lambda \in \mathbb{K}$ is a root of order p of $P \in \mathbb{K}[X]$ if and only if :

1) $\exists Q \in \mathbb{K}[X], P = (X-\lambda)^p Q$ with $Q(\lambda) \neq 0$.

This allows to prove that if λ is a root of order $p \geq 1$ of P, it is a root of order $p-1$ of P'.

Another characterisation for $P \neq 0$: *2)* $\forall k \in [\![0, p-1]\!], \quad P^{(k)}(\lambda) = 0$ and $P^{(p)}(\lambda) \neq 0$.

Too many roots

If a polynomial $P \in \mathbb{K}_n[X]$ has $n+1$ roots then $P = 0$

As a consequence, if a polynomial P has an infinite amount of roots then $P = 0$, and if P and Q have the same values at an infinite amount of points then $P = Q$.

Gauss's theorem

Let $P \in \mathbb{C}[X]$. Then P is totally separated.

Root-coefficient relations

Let $P = \sum_{k=0}^{n} a_k X^k$ with $a_n \neq 0$, we suppose $P = \lambda\prod_{i=0}^{n}(X - \lambda_i)$.

We have $\boxed{\sum_{i=1}^{n}\lambda_i = -\frac{a_{n-1}}{a_n} \text{ and } \prod_{i=1}^{n}\lambda_i = (-1)^n\frac{a_0}{a_n}}$

We have more generally for all $k \in [\![1, n]\!]$, $\quad \sum_{1 \leq i_1 < ... < i_k \leq n} \lambda_{i_1}...\lambda_{i_k} = (-1)^k\frac{a_{n-k}}{a_n}$

Example : let $P = aX^2 + bX + c$, we have $r_1 + r_2 = -\frac{b}{a}$ and $r_1 r_2 = \frac{c}{a}$.

8.4 Bases of $\mathbb{K}[X]$

Echelon families

If $(P_i)_{i\in I}$ is a family of nonzero polynomials of mutually distict degrees, it is independent.

Taylor's formula

Let $a \in \mathbb{K}$ and $P \in \mathbb{K}_n[X]$. We have $\boxed{P = \sum_{k=0}^{n} \frac{P^{(k)}(a)}{k!}(X-a)^k}$

Lagrange interpolation

Let $(x_0, ..., x_n)$ be mutually different scalars. We define the associated LAGRANGE polynomials :

$\forall k \in [\![0, n]\!]$, $\boxed{L_k = \prod_{\substack{i=0 \\ i\neq k}}^{n} \frac{X - x_i}{x_k - x_i}}$ They satisfy $\forall (k, j) \in [\![0, n]\!]^2, \quad L_k(x_j) = \delta_{k,j}$

$(L_k)_{0\leq k\leq n}$ is a basis of $\mathbb{K}_n[X]$ with $\forall P \in \mathbb{K}_n[X]$, $\boxed{P = \sum_{k=0}^{n} P(x_k)L_k}$

8.5 TD

Ex 41

GRAM's matrix. Let E be a real prehilbertian space and $(x_1, ..., x_n) \in E^n$. Let $G(x_i) = ((x_i|x_j))_{(i,j)\in[\![1,n]\!]^2}$ and $g(x_i) = \det G(x_i)$.

1) Find $A \in M_n(\mathbb{R})$ so that $G(e_i) = A^T A$. Use this to prove :

a) (x_i) is independent $\Leftrightarrow g(x_i) \neq 0$

b) $\text{rank}(x_i) = \text{rank}G$ (*Hint* : prove $\text{Ker}G = \text{Ker}A$)

2) Let F be a finite-dimensional subspace of E. Let $(e_1, ..., e_n)$ be a basis of F. Let $x \in E$. We define the **euclidian distance** between x and F $\text{d}(x, F) = \|x - p(x)\|$ were p is the orthogonal projector on F.

Prove that $\text{d}(x, F) = \sqrt{\dfrac{g(x, e_1, ..., e_n)}{g(e_1, ..., e_n)}}$

Ex 42

1) OT decomposition. Let $A \in GL_n(\mathbb{R})$. Show $\exists O \in O_n(\mathbb{R}), \quad \exists T \in T_n^+(\mathbb{R}), \quad A = OT$
Hint : use SCHIMDT's process.

2) HADAMARD's inequality : prove that $|\det A| \leq \prod_{j=1}^{n} \|C_j\|$

(where the C_j are the columns of A)

Ex 43

TCHEBYCHEV's polynomials.

1) Let $n \in \mathbb{N}$. Prove that $\exists! T_n \in \mathbb{R}[X]$ so that $\forall \theta \in \mathbb{R}, \quad T_n(\cos\theta) = \cos(n\theta)$

2) Find the dominant coefficient of T_n and its degree.

3) Prove that $T_{n+2} - 2XT_{n+1} + T_n = 0$.

4) Factorise T_n.

Ex 44

VANDERMONDE's matrix. Let $(x_1, ..., x_n) \in \mathbb{K}^n$. Consider :

$$V(x_i) = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{pmatrix} \text{. and } \quad \varphi : \begin{cases} \mathbb{K}_{n-1}[X] & \longrightarrow & \mathbb{K}^n \\ P & \longmapsto & \begin{pmatrix} P(x_1) \\ \vdots \\ P(x_n) \end{pmatrix} \end{cases}$$

1) Prove that φ is an isomorphism if and only if the (x_i) are mutually distinct.

We now consider $\mathbb{K} = \mathbb{R}$ and $(x_1, ..., x_n) = (1, ..., n)$.

2) Explain the link between φ and $V(1, ..., n)$ and use $\psi = \varphi^{-1}$ to compute $V(1, ..., n)^{-1}$

9 Calculus

9.1 Complements on sequences

Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$

$\varepsilon - \delta$ definitions

Let $(u_n) \in \mathbb{K}^{\mathbb{N}}$:

- $u_n \xrightarrow[n \longrightarrow +\infty]{} l \in \mathbb{K} \quad \Leftrightarrow \quad \forall \varepsilon > 0, \quad \exists N \in \mathbb{N}, \quad \forall n \geq N, \quad |u_n - l| \leq \varepsilon$
- If $\mathbb{K} = \mathbb{R}$, $u_n \xrightarrow[n \longrightarrow +\infty]{} +\infty \quad \Leftrightarrow \quad \forall M > 0, \quad \exists N \in \mathbb{N}, \quad \forall n \geq N, \quad u_n \geq M$

Monotonous convergence theorem

Let $(u_n) \in \mathbb{R}^{\mathbb{N}}$ a monotonous sequence. (u_n) has a limit in $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$.

Definition

An **extraction** φ is an injective map $\mathbb{N} \longrightarrow \mathbb{N}$.

If φ is an injection then $\forall n \in \mathbb{N}, \quad \varphi(n) \geq n$. If $u_n \longrightarrow l$ then $u_{\varphi(n)} \longrightarrow l$.

A **sub-sequence** of (u_n) is a sequence $(u_{\varphi(n)})$ where φ is an extraction.

Bolzano-Weierstrass Theorem in $\mathbb{R}$

All bounded sequences of $\mathbb{R}$ have a convergent subsequence.

9.2 Complements on functions

We consider a function $f : X \longrightarrow \mathbb{K}$ with $X \subset \mathbb{K}$.

Definition

Let $\alpha \in \mathbb{R}_+$. f is α-**lipschitzian** when $\forall (x,y) \in X^2, \quad |f(x) - f(y)| \leq \alpha |x-y|$

$\varepsilon - \delta$ definitions

Let $(a,b) \in X \times \mathbb{K}$.

We say that $f(x) \xrightarrow[x \to a]{} b$ when $\forall \varepsilon > 0, \quad \exists \delta > 0, \quad \forall x \in X, \quad |x-a| \leq \delta \Rightarrow |f(x) - b| \leq \varepsilon$

We also define limits at $\pm\infty$ and to $\pm\infty$ when $\mathbb{K} = \mathbb{R}$. For example :

$f(x) \xrightarrow[x \to -\infty]{} +\infty \quad \Leftrightarrow \quad \forall M > 0, \quad \exists m \in \mathbb{R}, \quad \forall x \leq m, \quad f(x) \geq M$

Sequential characterisation of the limit

Let $f : X \longrightarrow \mathbb{R}$, $a \in \overline{X}$, $b \in \overline{\mathbb{R}}$.

$$\boxed{f(x) \xrightarrow[x \to a]{} b \iff \forall (x_n) \in X^{\mathbb{N}} \text{ so that } x_n \xrightarrow[n \to +\infty]{} a, \quad f(x_n) \xrightarrow[n \to +\infty]{} b}$$

Monotonous limit theorem

Let $-\infty \leq u < v \leq +\infty$ and $f :]u,v[\longrightarrow \mathbb{R}$ monotonous.

Then f has a limit (in $\overline{\mathbb{R}}$) at v^- and a limit at u^+.

9.3 Continuity

Definition

Let $f : X \longrightarrow \mathbb{K}$. f is said to be **continuous** at $a \in X$ when $f(x) \xrightarrow[x \to a]{} f(a)$.

f is said to be **left-handedly continous** a a when $f(x) \xrightarrow[x \to a^-]{} f(a)$
(**right-handedly** : $f(x) \xrightarrow[x \to a^+]{} f(a)$)

If f is continuous at both sides of a then it is continous at point a.

f is said to be continous on X when it is continous at every point of X.

Other forms of continuity

f is said to be **uniformly continuous** when :
$\forall \varepsilon > 0, \quad \exists \delta > 0, \quad \forall a \in X, \quad \forall x \in [a-\delta, a+\delta], \quad |f(x) - f(a)| \leq \varepsilon$

f is lipschitzian $\Rightarrow$ f is uniformly continuous $\Rightarrow$ f is continuous.

Heine's theorem

Let $f : [a,b] \longrightarrow \mathbb{K}$ continuous. Since $[a,b]$ is a segment, f is uniformly continuous.

Intermidiate values theorem (IVT)

Let $f : [a,b] \longrightarrow \mathbb{R}$ a continuous function. f reaches all values between $f(a)$ and $f(b)$.

Reached bounds theorem

Let $f : [a, b] \longrightarrow \mathbb{R}$ continuous. Since $[a, b]$ is a segment, f reaches a minimum and a maximum.

9.4 Complements on Differentiation

Definition

Let $f \in F(I, \mathbb{K})$ and $a \in I$. We define the **slope** at a of f : $S_a : \begin{cases} I \setminus \{a\} & \longrightarrow & \mathbb{K} \\ x & \longmapsto & \dfrac{f(x) - f(a)}{x - a} \end{cases}$

When they exist, we define $f'(a) = \lim\limits_{x \longrightarrow a} S_a(x)$, $f'_l(a) = \lim\limits_{x \longrightarrow a^-} S_a(x)$, $f'_r(a) = \lim\limits_{x \longrightarrow a^+} S_a(x)$

Reciprocal bijection

Let f continuous and strictly monotonous on I, inducting a bijection from I to $J = f(I)$. Let $g : J \longrightarrow I$ its reciprocal bijection. Suppose that f is differentiable at a.

"g is differentiable at $b = f(a)$" $\iff f'(a) \neq 0$ In that case, $g'(b) = \dfrac{1}{f'(a)}$

Rolle's theorem

Let $a < b$, $f : [a, b] \longrightarrow \mathbb{R}$ continuous on $[a, b]$ and differentiable on $]a, b[$ so that $f(a) = f(b)$. Then $\exists c \in]a, b[, \quad f'(c) = 0$

Mean value theorem

Let $f : [a, b] \longrightarrow \mathbb{R}$ continuous and differentiable on $]a, b[$. Then $\exists c \in]a, b[, \quad f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

Mean inequality

Let $f : [a, b] \longrightarrow \mathbb{R}$ continuous and differentiable on $]a, b[$.
If $\exists K \geq 0$ so that $|f'| \leq K$, then f is K-lipschitzian.

9.5 Successive differentiation

Let I be an interval of $\mathbb{R}$.

Definition

$f : I \longrightarrow \mathbb{K}$ is said to be **of class** D^n when it is n times differentiable. We note its k-th derivatives $f^{(k)}$ ($k \in [\![0, n]\!]$). We write $f \in D^n(I, \mathbb{K})$.

$f : I \longrightarrow \mathbb{K}$ is said to be **of class** C^n when it is n times differentiable and $f^{(n)}$ is continuous. We write $f \in C^n(I, \mathbb{K})$. If f is infinitely differentiable, we write $f \in C^\infty(I, \mathbb{K})$.

Leibniz's formula

Let $(f, g) \in C^n(I, \mathbb{K})[$". Then fg is of class C^n with $\boxed{(fg)^{(n)} \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}}$

Taylor with integral remainder

Let $n \in \mathbb{N}$, $f \in C^{n+1}(I, \mathbb{K})$ and $a \in I$. We have :

$$\boxed{\forall x \in I, \quad f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)} \mathrm{d}t}$$

Taylor-Lagrange inequality

Let $f \in C^{n+1}(I, \mathbb{K})$, $a \in I$ and $x \in I$. Let M be an upper bound of $|f^{(n+1)}|$. We have :

$$\boxed{\left| f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \right| \leq M \frac{|x-a|^{n+1}}{(n+1)!}}$$

9.6 TD

Ex 45

Let $f \in C^0(\mathbb{R}_+, \mathbb{R}_+)$ so that $\dfrac{f(x)}{x} \xrightarrow[x \longrightarrow +\infty]{} l < 1$.
Prove that f has a fixed point α ($f(\alpha) = \alpha$).

Ex 46

Let $f \in C^n(\mathbb{R}, \mathbb{R})$ so that $f(0) = f(1) = ... = f(n)$. Prove that $f^{(n)}$ has a zero.

Ex 47

a) **Limit of the derivative theorem** : let $I \subset \mathbb{R}$ an interval, $a \in I$ and $f \in C^0(I)$ differentiable on $I \setminus \{a\}$ so that $f'(x) \xrightarrow[\substack{x \longrightarrow a \\ x \neq a}]{} \lambda \in \mathbb{R}$.

Prove that f is differentiable at a with $f'(a) = \lambda$

b) Application : let $f : \left\{ \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \left\{ \begin{array}{c} e^{-\frac{1}{x^2}} \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{array} \right. \end{array} \right.$

Prove that f is of class C^1 on $\mathbb{R}$.

Ex 48

Darboux's theorem : Let $f \in D^1([a, b], \mathbb{R})$ and y between $f'(a)$ and $f'(b)$. Prove that $\exists c \in]a, b[, \quad f'(c) = y$.

Hint : Consider the slopes at a and b.

Ex 49

1) **MVT generalisation** : Let $a < b$ two real numbers and $\varphi, \psi : [a, b] \longrightarrow \mathbb{R}$ continuous on $[a, b]$ and differentiable on $]a, b[$.
Prove that $\exists c \in]a, b[, \quad (\varphi(b) - \varphi(a))\psi'(c) = (\psi(b) - \psi(a))\varphi'(c)$

2) L'Hospital's rule : let $f, g : [a, b] \longrightarrow \mathbb{R}$ continuous on $[a, b]$ and differentiable on $]a, b[$, with $\forall x \in]a, b[, g'(x) \neq 0$, $f(a) = g(a) = 0$ and $\exists l \in \overline{\mathbb{R}}$: $\dfrac{f'(x)}{g'(x)} \xrightarrow[\substack{x \longrightarrow a \\ x \neq a}]{} l$

a) Prove that $\forall x \in]a, b[, \quad g(x) \neq 0$.

b) Use the first question to prove that $\dfrac{f(x)}{g(x)} \xrightarrow[\substack{x \longrightarrow a \\ x \neq a}]{} l$

10 Topology

10.1 Norms

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ a field and E a $\mathbb{K}$-vector space.

Definition

A **norm** is a map $N : E \longrightarrow \mathbb{R}_+$ verifying :

- **homogeneity** : $\forall(\lambda, x) \in \mathbb{K} \times E, \quad N(\lambda x) = |\lambda| N(x)$
- **triangualr inequality** : $\forall(x, y) \in E^2, \quad N(x+y) \leq N(x) + N(y)$
- **separation** $\forall x \in E, \quad N(x) = 0 \Rightarrow x = 0$

  We often note norms $\|\cdot\|$. Its associated **distance** is $d : \begin{cases} E^2 & \longrightarrow & \mathbb{R}_+ \\ (x,y) & \longmapsto & \mathrm{d}(x,y) = \|x-y\| \end{cases}$

Examples :

- $E = \mathbb{R}^n$: euclidian norm $\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$, infinite norm $\|x\|_\infty = \max_i |x_i|$
- $E = C^0([a,b])$: euclidian norm $\|f\|_2 = \sqrt{\int_a^b f^2}$, infinite norm : $\|f\|_\infty = \sup_{x \in [a,b]} |f(x)|$
- If N is a norm on E and $u \in L(E)$ is injective, then $N(u(\cdot))$ is a norm on E.

Definition

Let $a \in E$ and $r > 0$.

- The **open ball** of center a and radius r is $B_o(a,r) = \{x \in E | d(a,x) < r\}$
- The **closed ball** of center a and radius r is $B_c(a,r) = \{x \in E | d(a,x) \leq r\}$
- The **sphere** of center a and radius r is $S(a,r) = \{x \in E | d(a,x) = r\}$

Definition

Let $A \neq \varnothing \subset E$ and $x \in E$. The **distance** between x and A is $\mathrm{d}(x,A) = \inf\{\mathrm{d}(x,a) | a \in A\}$.

The **diameter** of A is $\mathrm{diam}(A) = \sup\{\mathrm{d}(a,b) | (a,b) \in A^2\}$

A is **bounded** when $\exists M > 0, \quad \forall a \in A, \quad \|x\| \leq M$

Let $u : E \longrightarrow F$. u is **bounded** when $\exists M > 0, \quad \forall x \in E, \quad \|u(x)\| \leq M$

Norm comparison

Let N_1 and N_2 be two norms on E.
N_1 is **dominated** by N_2 when $\exists \alpha \in \mathbb{R}_+^* : \quad N_1 \leq \alpha N_2$

N_1 and N_2 are **equivalent** when $\exists(\alpha, \beta) \in \left(\mathbb{R}_+^*\right)^2 : \quad \alpha N_1 \leq N_2 \leq \beta N_1$

10.2 Topology of a normed vector space

Adherence values

Let $(u_n) \in E^{\mathbb{N}}$. An **adherence value** of (u_n) is a limit of a subsequence of (u_n). Let $\lambda \in E$:

$$\lambda \text{ is an adherence value of } (u_n) \iff \forall \varepsilon > 0, \quad \forall n_0 \in \mathbb{N}, \quad \exists n \geq n_0, \quad \mathrm{d}(u_n, \lambda) \leq \varepsilon$$

Open parts

Let $O \subset E$. O is said to be **open** when $\forall x \in O, \quad \exists r > 0, \quad B_o(x, r) \subset O$.
Any union of open parts is open, a finite intersection of opens is open.

Closed parts

Let $C \subset E$. C is said to be **closed** when $E \setminus C$ is open.
A finite union of closed parts is closed, any intersection of closed parts is open.

Vicinities

Let $a \in E$. V is a **vicinity** of a when $\exists r > 0, \quad B_o(a, r) \subset V$.
The set of the vicinities of a is written $\mathcal{V}_a$.

Interior

Let $A \subset E$, and $a \in E$. a is an **interior point** of A if A is a vicinity of a.

the **interior** of A, written $\mathring{A}$ or $\mathrm{Int}(A)$ is the biggest open part included in A.

$\mathring{A}$ is also the set of the interior points of A.

Closure

Let $A \subset E$ and $a \in A$. a is **adherent** to A if $\forall r > 0, \quad B_o(a, r) \cap A \neq \varnothing$

The **closure** of A, written $\overline{A}$ or $C(A)$ is the smallest closed set containing A.

$\overline{A}$ is also the set of all the adherent points to A.

$\overline{A}$ is the set of the limits of the convergent sequences of A.

A is closed $\Leftrightarrow$ all convergent sequences of A converge in A.

Definition

Let $A \subset E$ and $D \subset A$. D is said to be **dense** in A if one of the following equivalent properties is met :

1) $A \subset \overline{D}$

2) $\forall a \in A, \quad \forall r > 0, \quad \exists x \in D, \quad \mathrm{d}(x, a) \leq r$

3) $\forall a \in A, \quad \exists (d_n) \in D^{\mathbb{N}}, \quad d_n \longrightarrow a$

Definition

Let $A \subset E$. The **boundary** of A, written ∂A is $\partial A = \overline{A} \setminus \mathring{A}$.

10.3 Continuity

Let E, F be two normed vector spaces and $A \subset E$.

Definition

Let $f : A \longrightarrow F, a \in A, l \in F$. We say that $f \underset{a}{\rightarrow} l$ when one of the two equivalent properties is met :

- $\forall V \in \mathcal{V}_l, \quad \exists W \in \mathcal{V}_a, \quad f(W \cap A) \subset V$
- $\forall (a_n) \in A^{\mathbb{N}}$ so that $a_n \longrightarrow a$, $f(a_n) \longrightarrow l$

Example $f : \begin{cases} \mathbb{R}^2\} & \longrightarrow & \mathbb{R} \\ (x,y) & \longmapsto & \dfrac{xy}{x^2+y^2} \text{ if } (x,y) \neq (0,0), 0 \text{ otherwise} \end{cases}$ is not continuous at $(0,0)$

Definition

$f : A \longrightarrow F$ and $k \leq 0$. f is k-**lipschitzian** when $\forall (x,y) \in A^2, \quad \mathrm{d}(f(x), f(y)) \leq k \; \mathrm{d}(x,y)$

If f is linear then it is k-lipschitzian iif $\forall x \in A, \quad \|u(x)\| \leq \|x\|$

Image by a continuous map

Let $f : A \longrightarrow F$ continuous. Let $x \in A$, $V_{f(x)} \in \mathcal{V}_{f(x)}$, O an open of A and C a closed of A.

- $f^{-1}\left(V_{f(x)}\right) \in \mathcal{V}_x$
- $f^{-1}(O)$ is open
- $f^{-1}(C)$ is closed

Definition

$f : A \longrightarrow F$ is **uniformly** continuous when :

$\forall \varepsilon > 0, \quad \exists \eta > 0, \quad \forall (x,y) \in A^2, \quad \mathrm{d}(x,y) \leq \eta \Rightarrow \mathrm{d}(f(x), f(y)) \leq \varepsilon$

Continuity of linear maps

Let $u \in L(E,F)$. u is continuous $\Leftrightarrow \exists k \geq 0, \quad \forall x \in E, \quad \|u(x)\| \leq k \|x\|$

10.4 Compacity

Let E be a normed vector space.

Definition

A set $K \subset E$ is **compact** when all sequences of K have an adherence value in K.

All compacts are closed and bounded.

If K is compact and C is a closed subset of K then C is compact.

Bolzano-Weiestrass Theorem

The compacts of $\mathbb{R}^n$ are the parts that are closed and bounded.

A criteria for convergene

Let K a compact. If $(u_n) \in K^{\mathbb{N}}$ has only one adherence value l, then $u_n \longrightarrow l$.

Image of a compact by a continuous map

Let K a compact and $f : K \longrightarrow F$ continuous. Then $f(K)$ is a compact.

Reached bounds theorem

Let K a compact and $f \in C^0(K, \mathbb{R})$. Then f is bounded and reaches its bounds.

Heine's theorem

Let K a compact and $f \in C^0(K, F)$. Then f is uniformously continuous.

Norm equivalence

Suppose E of finite dimension. Then all norms of E are equivalent.

Proof

1) Let $(e_1, ..., e_n)$ a basis of E. We define the norms :

N_∞ on E : $\left\{ \begin{array}{ccc} E & \longrightarrow & \mathbb{R}_+ \\ x = \sum_{i=1}^{n} x_i e_i & \longmapsto & \max_i |x_i| \end{array} \right.$ and $\|\cdot\|_\infty$ on $\mathbb{K}^n$: $\left\{ \begin{array}{ccc} \mathbb{K}^n & \longrightarrow & \mathbb{R}_+ \\ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} & \longmapsto & \max_i |x_i| \end{array} \right.$

The map f : $\left\{ \begin{array}{ccc} \mathbb{K}^n & \longrightarrow & E \\ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} & \longmapsto & \sum_{i=1}^{n} x_i e_i \end{array} \right.$ is an isometry from $(\mathbb{K}^n, \|\cdot\|_\infty)$ to (E, N_∞).

Since $\forall x \in E, \quad N_\infty(f(x)) = \|x\|_\infty$, f is continuous.

Since $K = S(0,1)_{\mathbb{K}^n}$ is compact and f is continuous, $\boxed{f(K) = S(0,1)_E \text{ is compact}}$

2) Let N a norm on E, we shall prove that N and N_∞ are equivalent.

Let $x = \sum_{i=1}^{n} x_i e_i \in E$. $N(x) = N\left(\sum_{i=1}^{n} x_i e_i\right) \leq \left(\sum_{i=1}^{n} N(e_i)\right) N_\infty(x)$ Let $\alpha = \sum_{i=1}^{n} N(e_i) > 0$.

We have $\boxed{N \leq \alpha N_\infty}$ So N is continuous on (E, N_∞) :

$\forall (x,y) \in E^2, \quad |N(x) - N(y)| \leq N(x-y) \leq \alpha N_\infty(x-y)$ so N is α-lipschitzian so continuous.

Then N is continuous on the compact $S = S(0,1)_{(E,N_\infty)}$:

so $m = \inf_S N$ is reached at a certain $x \in S$: $m = N(x) > 0$. (reached bounds theorem)

Therefore $\forall x \in S, \quad N(x) \geq m$

Let $x \in E \setminus \{0\}$, and $u = \dfrac{x}{N_\infty(x)} \in S$. $N(x) = N(N_\infty(x)u) = N_\infty(x)N(u) \geq N_\infty(x) \times m$

Therefore $\boxed{\forall x \in E, \quad mN_\infty \leq N \leq \alpha N_\infty}$ so N and N_∞ are equivalent.

Continuity of linear maps in finite-dimensional spaces

Let $u \in L(E, F)$ where E is finite-dimensional. Then u is continuous.

10.5 TD

Ex 50

Let $f \in C^1(\mathbb{R},\mathbb{R})$ and let $F : \begin{cases} \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (x,y) & \longmapsto & \dfrac{f(x)-f(y)}{x-y} \text{ if } x \neq y, \quad f'(x) \text{ otherwise} \end{cases}$.

Prove that $F \in C^0(\mathbb{R}^2)$.

Ex 51

Subordinate norms. Let E and F two normed vector spaces. We note $L_c(E,F)$ the vector space of the continuous linear maps from E to F. Let $u \in L_c(E,F)$.

We consider $A = \{k \in \mathbb{R}_+ : \forall x \in E, \quad \|u(x)\| \leq k\|x\|\}$ and $|||u||| = \inf A$.

1) Prove that $\forall x \in E, \quad \|u(x)\| \leq |||u|||\|x\|$

2) Prove that $|||u||| = \sup\limits_{x\neq 0} \dfrac{\|u(x)\|}{\|x\|} = \sup\limits_{\|x\|=1} \|u(x)\|$

3) Prove that $|||\cdot|||$ is a norm on $L_c(E,F)$.

4) Let G another normed vector space, $u \in L_c(E,F)$ and $v \in L_c(F,G)$.

Prove that $|||v \circ u||| \leq |||u||| |||v|||$.

Ex 52

Two classics :

1) Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$, $M \in M_n(\mathbb{K})$ and $P = \det(M - XI_n)$

Noticing that M is inversible $\Leftrightarrow P(0) \neq 0$ and find $(A_n) \in GL_n(\mathbb{K})^{\mathbb{N}}$ so that $A_n \longrightarrow M$

2) Prove that $O_n(\mathbb{R})$ is a compact.

Ex 53

Let K a compact and $C_1 \supset C_2 \supset \ldots$ a sequence of non-empty closed parts of K that are fitted together.

Prove that $\bigcap\limits_{n\in\mathbb{N}} C_n \neq \varnothing$.

11 Taylor Expansion

Definition

- We say that $u_n = o(v_n)$ when after a certain rank we can write $u_n = v_n w_n$ with $w_n \longrightarrow 0$.

  If $v_n \neq 0$ after a certain rank, another definition is $\dfrac{u_n}{v_n} \longrightarrow 0$
- We say that $u_n = O(v_n)$ when after a certain rank we can write $u_n = v_n w_n$ where (w_n) is bounded.

  If $v_n \neq 0$ after a certain rank, another definition is $\left(\dfrac{u_n}{v_n}\right)$ is bounded after a certain rank.
- We say that $u_n \sim v_n$ when after a certain rank we can write $u_n = v_n w_n$ with $w_n \longrightarrow 1$.

  If $v_n \neq 0$ after a certain rank, another definition is $\dfrac{u_n}{v_n} \longrightarrow 1$

  $\sim$ is an equivalence relation.

Let $f : X \longrightarrow \mathbb{K}$ and $a \in \overline{X}$

Definition

- We say that $f(x) = o(g(x))$ at the vicinity of a when at the vicinity of a we can write $f(x) = g(x)h(x)$ where $h(x) \xrightarrow[x \longrightarrow a]{} 0$.

  When $g(x) \neq 0$ at the vicinity of a, another definition is $\dfrac{f(x)}{g(x)} \xrightarrow[x \longrightarrow a]{} 0$
- We say that $f(x) = O(g(x))$ at the vicinity of a when at the vicinity of a we can write $f(x) = g(x)h(x)$ with h bounded.

  When $g(x) \neq 0$ at the vicinity of a, another definition is $\dfrac{f}{g}$ is bounded at the vicinity of a.
- We say that $f(x) \sim g(x)$ at the vicinity of a when at the vicinity of a we can write $f(x) = g(x)h(x)$ where $h(x) \xrightarrow[x \longrightarrow a]{} 1$.

  When $g(x) \neq 0$ at the vicinity of a, another definition is $\dfrac{f(x)}{g(x)} \xrightarrow[x \longrightarrow a]{} 1$

Compared growths

Let $\alpha, \beta > 0$. We have when $x \longrightarrow +\infty$, $\ln^\alpha(x) = o(x^\beta)$ and $x^\alpha = o\left(e^{\beta x}\right)$

When $n \longrightarrow +\infty$, we have : $e^{\alpha n} = o(n!)$ and $n! = o(n^n)$.

Logarithmic comparison

Let (u_n) and (v_n) two strictly positive sequences so that ACR, $\dfrac{u_{n+1}}{v_{n+1}} \leq \dfrac{u_n}{v_n}$. Then $u_n = O(v_n)$

Finding limits :

- $\dfrac{(1+\frac{1}{n})^{1+\frac{1}{n}} - 1}{\sin \frac{1}{n}}$
- $\left(\dfrac{\ln(1+x)}{\ln(x)}\right)^x$ when $x \longrightarrow +\infty$

Taylor expansion

f has a Taylor expansion at the order n at the point a ("$TE_n(a)$") when there exists $(c_0, ..., c_n) \in \mathbb{K}^{n+1}$ such that at the vicinity of a :

$$f(x) = \sum_{k=0}^{n} c_k (x-a)^k + o\left((x-a)^n\right)$$

$x \longmapsto \dfrac{1}{1-x}$ has a $TE_0(n)$ for all n : $\dfrac{1}{1-x} = \displaystyle\sum_{k=0}^{n} x^k + o(x^n)$.

Characterisations for $n = 0, 1$

f has a $TE_0(a) \Leftrightarrow f$ is continuous at a. f has a $TE_1(a) \Leftrightarrow f$ is differentiable at a.

Properties

- Taylor expansions are unique.
- If a function is even, its odd terms are 0. If it is odd then its even terms are 0.
- If f, g have a $TE_n(a)$ then their linear combinations and product also have a $TE_n(a)$.

Taylor expansion of an antiderivative

Let $f : I \longrightarrow \mathbb{K}$ admitting a $TE_n(a)$: $f(x) = \sum_{k=0}^{n} c_k (x-a)^k + o\left((x-a)^n\right)$

Let F an antiderivative of f. Then $F(x) = F(a) + \sum_{k=0}^{n} \frac{c_k}{k+1}(x-a)^{k+1} + o\left((x-a)^{n+1}\right)$

Taylor-Young formula

Let $f \in C^n(I, \mathbb{K})$, and $a \in I$. Then f has a $TE_n(a)$ given by :

$\boxed{f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + o\left((x-a)^n\right)}$ for $h \longrightarrow 0$: $f(a+h) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} h^k + o(h^n)$

$$\frac{1}{1-x} = 1 + x + x^2 + \ldots + x^n + o(x^n)$$

$$\frac{1}{1+x} = 1 - x + x^2 - \ldots + (-x)^n + o(x^n)$$

$$\forall \alpha \in \mathbb{C},\ (1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{6}x^3 + \ldots + \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!}x^n + o(x^n)$$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots + \frac{x^n}{n!} + o(x^n)$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + (-1)^n \frac{x^k}{k} + o(x^n)$$

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + o(x^{2n+1})$$

$$\sin(x) = x - \frac{x^3}{6} + \frac{x^5}{120} - \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + o(x^{2n+2})$$

$$\mathrm{ch}(x) = 1 + \frac{x^2}{2} + \frac{x^4}{24} + \ldots + \frac{x^{2n}}{(2n)!} + o(x^{2n+1})$$

$$\mathrm{sh}(x) = x + \frac{x^3}{6} + \frac{x^5}{120} + \ldots + \frac{x^{2n+1}}{(2n+1)!} + o(x^{2n+2})$$

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots (-1)^{2n+1} \frac{x^{2n+1}}{2n+1} + o(x^{2n+2})$$

$$\tan(x) = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + o(x^7)$$

12 Uniform Convergence

Let E and F two normed vector spaces and $A \subset E$. Let (f_n) a sequence of maps from A to F.

Simple convergence

We say that (f_n) **converges simply (CVS)** towards f when : $\forall a \in A, \quad f_n(a) \longrightarrow f(a)$

This can be written : $\forall a \in A, \quad \forall \varepsilon > 0, \quad \exists n_a \in \mathbb{N}, \quad \forall n \geq n_a, \quad \|f_n(a) - f(x)\| \leq \varepsilon$

Uniform Convergence

Let $X \subset A$. We say that (f_n) **converges uniformly (CVU)** on X towards f when :

$\forall \varepsilon > 0, \quad \exists N \in \mathbb{N}, \quad \forall n \geq N, \quad \forall x \in X, \quad \|f_n(x) - f(x)\| \leq \varepsilon$

Other definition : $\sup\limits_{x \in X} \|f_n(x) - f(x)\| \xrightarrow[n \longrightarrow +\infty]{} 0$ (With $f_n - f$ bounded on X)

Other definition : There exists $(\alpha_n) \in \mathbb{R}_+^{\mathbb{N}}$ such that ACR, $\|f_n(x) - f(x)\| \leq \alpha_n$ and $\alpha_n \longrightarrow 0$

Uniform convergence on A for bounded functions $A \longrightarrow F$ is the same as convergence in the normed vector space $(\mathcal{B}(A, F), N_\infty)$

Lemma

Let $a \in \overline{A}$, and (f_n) CVU on A towards f and such that $\forall n \in \mathbb{N}, \quad f_n(x) \xrightarrow[x \longrightarrow a]{} l_n \in F$.

If (l_n) converges in F, or $f(x)$ converges in F when $x \longrightarrow a$, then the other limit exists, and :

$\lim\limits_{n \to +\infty} \lim\limits_{x \to a} f_n(x) = \lim\limits_{x \to a} \lim\limits_{n \to +\infty} f_n(x)$

Consequence : if f is a uniform limit of continuous functions then it is continuous.

Double limit theorem

We suppose here that F is finite-dimensional.

Let (f_n) CVU towards f and such that $\forall n \in \mathbb{N}, \quad f_n(x) \xrightarrow[x \longrightarrow a]{} l_n \in F$.

Then (l_n) converges, f has a limit at a and both limits are the same.

For series

Consider a sequence (u_n) of bounded **functions** from A to F.

We say that $\sum u_n$ **converges normally (CVN)** when the series $\sum N_\infty(u_n)$ converges.

If a series converges normally and F is finite-dimensional, then $\sum u_n$ CVU on A.

Limit under the integral

Consider a sequence of functions (f_n) on a segment $[a,b]$ such that $f_n \xrightarrow[n\longrightarrow+\infty]{\text{(CVU)}} f$

Then f is continuous and $\int\limits_a^b f_n(t)\mathrm{d}t \xrightarrow[n\longrightarrow+\infty]{} \int\limits_a^b f(t)\mathrm{d}t$

For series, if $\sum u_n$ CVU on $[a,b]$ with each u_n continuous on $[a,b]$, then :

The sum is continuous, the series of the integrals converges and $\int\limits_a^b \sum\limits_{n=0}^{+\infty} u_n(t)\mathrm{d}t = \sum\limits_{n=0}^{+\infty} \int\limits_a^b u_n(t)\mathrm{d}t$

Uniformity and antiderivation

Let (f_n) a sequence of continuous functions on I CVU on all segments of I towards f. Let $a \in I$ and g_n the antiderivative of f_n such that $g_n(a) = 0$.

Then (g_n) CVU on all segments of I towards g, the antiderivative of f such that $g(a) = 0$.

Uniformity and differentiation

Let $(g_n) \in C^1(I)^{\mathbb{N}}$ CVS towards g, and with (g_n') CVU on all segments of I towards f.

Then g is of class C^1 on I, $g' = f$, and (g_n) CVU on all segments of I towards g.

To prove that a limit f of $(f_n) \in C^p(I)^{\mathbb{N}}$ is itself of class C^p, check :

- $\forall k \in [\![0, p-1]\!], \left(f_n^{(k)}\right)$ CVS on I
- $\left(f_n^{(p)}\right)$ CVU on all segments of I.

Differentiating a series

Let $\sum u_n$ a series of C^1 functions CVS on I. If $\sum u_n'$ CVU on all segments of I, then :

$\forall t \in I, \quad \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\sum\limits_{n=0}^{+\infty} u_n(t)\right) = \sum\limits_{n=0}^{+\infty} u_n'(t)$ and $\sum u_n$ CVU on all segments of I.

To prove that a sum $\sum u_n$ of C^p functions is itself of class C^p, check :

- $\forall k \in [\![0, p-1]\!], \sum u_n^{(k)}$ CVS on I
- $\sum u_n^{(p)}$ CVU on all segments of I.

12.1 TD

Ex 54

For $n \in \mathbb{N}$ we study $(E_n) : e^x + x = n$.

1) Prove that for all n in $\mathbb{N}$, (E_n) has a unique solution on $\mathbb{R}$ called u_n.

2) Prove $u_n \longrightarrow +\infty$

3) Prove $u_n \sim \ln(n)$

4) Studying $v_n = u_n - \ln(n)$, prove $u_n = \ln(n) - \dfrac{\ln(n)}{n} + o\left(\dfrac{\ln(n)}{n}\right)$

Ex 55

Compute $I(r,z) = \displaystyle\int_0^{2\pi} \frac{\mathrm{d}t}{z - re^{it}}$ for $(r,z) \in \mathbb{R}_+^* \times \mathbb{C}^*$ with $r \neq |z|$.

Hint : use geometric expansion

Ex 56

Let $f : \begin{cases} \mathbb{R}_+^* & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \displaystyle\sum_{n=1}^{+\infty} \frac{1}{1+n^2x} \end{cases}$

1) Prove that $f \in \mathcal{C}^\infty(\mathbb{R}_+^*)$

2) Prove that when $x \longrightarrow +\infty$, $f(x) \sim \dfrac{\pi^2}{6x}$ $\quad \left(\displaystyle\sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{6}\right)$

Ex 57

Study the convergence and uniform convergence of (f_n)

where $\forall n \in \mathbb{N}^*$, $\quad f_n : \begin{cases} \mathbb{R}_+ & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \left(1 - \dfrac{x}{n}\right)^n \text{ if } x \in [0,n], \quad 0 \text{ otherwise} \end{cases}$

13 Series

Definitions on series

Consider a sequence (u_n). It is the **term** of the series $\sum u_n$.

The **partial sums** of $\displaystyle\sum_{n \geq n_0} u_n$ are the $S_N = \displaystyle\sum_{n=n_0}^{N} u_n$.

The the sequence (S_N) converges, we say that the series converges and we define its **sum** $\displaystyle\sum_{n=n_0}^{+\infty} u_n$

The **remainder** of a convergent series is $R_N = \displaystyle\sum_{n=N+1}^{+\infty} u_n$

A series **converges absolutely (CVA)** when $\sum |u_n|$ converges. This implies simple convergence.

13.1 Summability

Definition

Let I a set and $(a_i)_{i\in I} \in \mathbb{K}^I$. We say that (a_i) is **summable** when $\sup\limits_{J \text{ finite } \subset I} \left(\sum\limits_{i\in J} |a_i|\right) \in \mathbb{R}$.

If the a_i are positive, one can always write $\sum\limits_{i\in I} a_i \in \mathbb{R} \cup \{+\infty\}$

Group summation for positive terms

Let $(a_i)_{i\in I}$ **positive real numbers**, and $(I_\lambda)_{\lambda\in\Lambda}$ a partition of I. Then : $\sum\limits_{i\in I} a_i = \sum\limits_{\lambda\in\Lambda} \left(\sum\limits_{i\in I_\lambda} a_i\right)$

And $(a_i)_{i\in I}$ is summable $\Leftrightarrow$ every $(a_i)_{i\in I_\lambda}$ is summable of sum s_λ **and** $(s_\lambda)_{\lambda\in\Lambda}$ is summable.

Group summation

Let $(a_i)_{i\in I}$ **a summable family**, and $(I_\lambda)_{\lambda\in\Lambda}$ a partition of I. Then : $\sum\limits_{i\in I} a_i = \sum\limits_{\lambda\in\Lambda} \left(\sum\limits_{i\in I_\lambda} a_i\right)$

And every $(a_i)_{i\in I_\lambda}$ is summable of sum s_λ with $(s_\lambda)_{\lambda\in\Lambda}$ summable.

Example : $\sum\limits_{n=1}^{+\infty} \frac{(-1)^n}{n^2}$

Fubini's Theorem

$(a_{p,q})_{(p,q)\in\mathbb{N}^2}$ is summable $\Leftrightarrow$ one of the 3 following properties is satisfied :

1) For all $p \in \mathbb{N}$, the series $\sum\limits_q |a_{p,q}|$ converges and has a sum s_p such that $\sum\limits_p s_p$ converges.

2) For all $q \in \mathbb{N}$, the series $\sum\limits_p |a_{p,q}|$ converges and has a sum s_q such that $\sum\limits_q s_q$ converges.

3) The series of term $s_n = \sum\limits_{p+q=n} |a_{p,q}|$ converges.

In this case, $\sum\limits_{(p,q)\in\mathbb{N}^2} a_{p,q} = \sum\limits_{p=0}^{+\infty}\sum\limits_{q=0}^{+\infty} a_{p,q} = \sum\limits_{q=0}^{+\infty}\sum\limits_{p=0}^{+\infty} a_{p,q} = \sum\limits_{n=0}^{+\infty}\sum\limits_{p+q=n} a_{p,q}$

Example : $\sum\limits_{(p,q)\in\mathbb{N}^2} \frac{(-1)^{p+q}}{2^p 3^q (p+q+1)}$

Cauchy product

Let $\sum a_n$ and $\sum b_n$ two absolutely convergent series.

Then the series of term $u_n = \sum\limits_{p+q=n} a_p b_q$ CVA and $\left(\sum\limits_{n=0}^{+\infty} a_n\right)\left(\sum\limits_{n=0}^{+\infty} b_n\right) = \sum\limits_{n=0}^{+\infty}\left(\sum\limits_{p+q=n} a_p b_q\right)$

13.2 Series convergence criteria

Comparing terms

Let (u_n) and (v_n) two complex sequence with (v_n) **positive**.

- If $\sum v_n$ converges and $u_n = O(v_n)$ then $\sum u_n$ CVA (same with o)
- If $u_n \sim v_n$ then $\sum u_n$ and $\sum v_n$ have the same nature.
- If after a certain rank $\dfrac{u_{n+1}}{u_n} \leq \dfrac{v_{n+1}}{v_n}$ with $u_n, v_n > 0$ then $u_n = O(v_n)$.

Riemann's reference series

$\displaystyle\sum_{n\geq 1} \frac{1}{n^\alpha}$ converges $\Leftrightarrow \alpha > 1$.

Method :

- If $\exists \alpha > 1, \quad n^\alpha u_n \longrightarrow 0$ then $\sum u_n$ converges.
- If $nu_n \longrightarrow +\infty$ then $\sum u_n$ diverges.

Example : $\displaystyle\sum \frac{\ln^\beta(n)}{n^\alpha}$

D'Alembert's rule

If $u_n > 0$ after a certain rank and $\dfrac{u_{n+1}}{u_n} \longrightarrow l$:

If $l < 1$ then $\sum u_n$ converges. If $l > 1$ then $\sum u_n$ diverges.

Example : $\displaystyle\sum \frac{x^n}{n!}$

Comparing with an integral

Let f piece-wise continuous, positive and decreasing on $[n_0, +\infty[$.

Then the series $\displaystyle\sum_{n\geq n_0+1} \left(\int_{n-1}^{n} f(t)\mathrm{d}t \; - f(n) \right)$ converges.

Examples : 1) $\displaystyle\sum_{k=1}^{n} \frac{1}{k^\alpha} \sim \frac{n^{1-\alpha}}{1-\alpha}$ $(\alpha < 1)$, and $\displaystyle\sum_{k=n+1}^{+\infty} \frac{1}{k^\alpha} \sim \frac{1}{(\alpha-1}n^{\alpha-1}$ when $\alpha > 1$, 2) $\displaystyle\sum_{n\geq 2} \frac{1}{n\ln^\alpha(n)}$

Alternated series criteria

If $u_n \longrightarrow 0$, $|u_n|$ decreases and $(-1)^n u_n$ is of constant sign,

Then $\sum u_n$ converges and $\forall n \in \mathbb{N}, \quad |R_n| \leq |u_{n+1}|$

Example : $u_n = \dfrac{(-1)^n}{n^\alpha + (-1)^n}$ with $\alpha > 0$

Slice summation

Let φ an extraction with $\varphi(0) = 0$. Let $v_n = \sum_{p=\varphi(n)}^{\varphi(n+1)-1} u_p$.

If $\sum_{p=\varphi(n)}^{\varphi(n+1)-1} |u_p| \xrightarrow[n \longrightarrow +\infty]{} 0$, and $\sum v_n$ converges, then $\sum u_p$ converges with $\sum_{n=0}^{+\infty} v_n = \sum_{p=0}^{+\infty} u_p$

Example : $\sum_{n \geq 1} \frac{e^{\frac{2in\pi}{3}}}{n}$

Antiderivative technique : if you study $\sum f(n)$, consider F an antiderivative of f and try to prove that the series of term $f(n) - (F(n+1) - F(n))$ converges absolutely.
That way, $\sum f(n)$ converges $\Leftrightarrow F(n)$ has a finite limit.

Example : $\sum_{n \geq 1} \frac{e^{i\sqrt{n}}}{n}$

13.3 Summing comparisons

Convergent case

Let (u_n) and (v_n) two complex sequences.

$\boxed{\text{If } v_n \text{ is the } \textbf{positive} \text{ term of a } \textbf{convergent} \text{ series}}$:

- $u_n = O(v_n) \Rightarrow R_n(u) = O(R_n(v))$
- $u_n = o(v_n) \Rightarrow R_n(u) = o(R_n(v))$
- $u_n \sim v_n \Rightarrow R_n(u) \sim R_n(v)$

Examples : $u_k = \frac{\ln(k+1) - \ln(k)}{k}$, $\quad \sum_{k=n+1}^{+\infty} \frac{k}{2^k}$

Divergent case

Let (u_n) and (v_n) two complex sequences.

$\boxed{\text{If } v_n \text{ is the } \textbf{positive} \text{ term of a } \textbf{divergent} \text{ series}}$:

- $u_n = O(v_n) \Rightarrow S_n(u) = O(S_n(v))$
- $u_n = o(v_n) \Rightarrow S_n(u) = o(S_n(v))$
- $u_n \sim v_n \Rightarrow S_n(u) \sim S_n(v)$

Examples : Cesaro summation, inductive sequence study : $u_{n+1} = \frac{1}{2}\text{Arctan}(u_n)$

13.4 TD

Ex 58

Harmonic series :

Let $H_n = \sum_{k=1}^{n-1} \frac{1}{k}$ $(n \geq 2)$. Prove that $\exists \gamma \in \mathbb{R}, \quad H_n = \ln(n) + \gamma - \frac{1}{2n} + o\left(\frac{1}{n}\right)$

Ex 59

Consider the sequence defined by $u_0 \in]0, \frac{\pi}{2}]$ and $\forall n \in \mathbb{N}, \quad u_{n+1} = \sin(u_n)$

1) Prove $u_n \longrightarrow 0$.

2) Let $(v_n) \in (\mathbb{R}_+^*)^{\mathbb{N}}$ and $\alpha \in \mathbb{R}$ such that $\frac{1}{v_{n+1}^\alpha} - \frac{1}{v_n^\alpha} \longrightarrow l > 0$. Find an equivalent of v_n.

3) Using *2)*, prove that $u_n \sim \sqrt{\frac{3}{n}}$

4) Find the second term in the asymptotical expansion of $\frac{1}{u_n^2}$.

Ex 60

Raabe-Duhamel criteria :

Let (u_n) a strictly positive real sequence such that $\frac{u_{n+1}}{u_n} = 1 - \frac{\alpha}{n} + o\left(\frac{1}{n}\right)$.

1) Comparing with $v_n = n^{-\beta}$, prove that if $\alpha > 1$, then $\sum u_n$ converges, and that it diverges if $\alpha < 1$.

2) We now suppose $\frac{u_{n+1}}{u_n} = 1 - \frac{\alpha}{n} + O\left(\frac{1}{n^2}\right)$. Prove that $\exists K > 0, \quad u_n \sim \frac{K}{n^\alpha}$

Ex 61

Abel transformation :

Let $(a_k), (b_k) \in \mathbb{C}^{\mathbb{N}}$, and (A_k) such that $\forall k \in \mathbb{N}, \quad a_k = A_k - A_{k-1}$.

1) Prove that for all $p \leq q$, we have $\sum_{k=p}^{q} a_k b_k = A_q b_q - A_{p-1} b_p + \sum_{k=p}^{q-1} A_k (b_k - b_{k+1})$

2) Study $\sum_{n \geq 1} \frac{e^{in\theta}}{n^\alpha}$ with $\theta \in \mathbb{R}, \alpha > 0$

14 Integration

Reminder : fundamental theorem of integration

Let $f \in C^0(I, \mathbb{K})$ and $a \in I$. $F : \begin{cases} I & \longrightarrow & \mathbb{K} \\ x & \longrightarrow & \int_a^x f(t)\mathrm{d}t \end{cases}$ is **the** antiderivative of f with $F(a) = 0$.

Definition

$f : [a,b] \longrightarrow \mathbb{K}$ is said to be **piecewise continuous (PWC)** when there exists $(a_0, ..., a_n)$ such that $a = a_0 < ... < a_n = b$ with :

for each $i \in [\![1,n]\!]$, f is continuous on $]a_{i-1}, a_i[$ and $f(x) \xrightarrow[x \longrightarrow a_{i-1}^+]{} \lambda \in \mathbb{K}$, $f(x) \xrightarrow[x \longrightarrow a_i^-]{} \mu \in \mathbb{K}$

A function is said to be piecewise continuous on an interval I if it is PWC on every segment of I. We write $f \in PWC(I)$

14.1 Generalised integrals on $[a, +\infty[$

Let $a \in \mathbb{R}$ and $f \in PWC([a, +\infty[)$

Definition

We say that $\int_a^{+\infty} f(t)\mathrm{d}t$ **converges** when $\lim_{x \longrightarrow +\infty} \int_a^x f(t)\mathrm{d}t$ exists.

In case of convergence, all usual integration theorems apply.

If f is positive and $\int_a^{+\infty} f(t)\mathrm{d}t$ diverges, we write $\int_a^{+\infty} f(t)\mathrm{d}t = +\infty$.

Comparisons

Let $(f,g) \in PWC([a,+\infty[)^2$ **positive** functions.

If $f = O(g)$ at $+\infty$, then the convergence of $\int_a^{+\infty} g(t)\mathrm{d}t$ implies the convergence of $\int_a^{+\infty} f(t)\mathrm{d}t$.

If $f \sim g$ at $+\infty$, then $\int_a^{+\infty} f(t)\mathrm{d}t$ and $\int_a^{+\infty} g(t)\mathrm{d}t$ are of the same nature.

Riemann comparison

Let $\alpha \in \mathbb{R}$. $\int_1^{+\infty} \frac{\mathrm{d}t}{t^\alpha}$ converges $\Leftrightarrow \alpha > 1$

Let $f \in PWC([a,+\infty[)$ a **positive** function.

If $\exists \alpha > 1$ so that $f(t) = O\left(\frac{1}{t^\alpha}\right)$ at $+\infty$, then $\int_a^{+\infty} f(t)\mathrm{d}t$ converges.

If $tf(t) \xrightarrow[t \longrightarrow +\infty]{} +\infty$, then $\int_a^{+\infty} f(t)\mathrm{d}t$ diverges.

Examples : $\int_1^{+\infty} t^\alpha e^{-t}$, $\int_2^{+\infty} \frac{\mathrm{d}t}{t^\alpha \ln^\beta(t)}$

Integrability

$f \in PWC([a,+\infty[)$ is **integrable** when $\int_a^{+\infty} |f(t)\mathrm{d}t$ converges. In that case, $\int_a^{+\infty} f(t)\mathrm{d}t$ converges.

You can also say that the integral converges absolutely.
When there is only one interval with only one improper bound, you can just say "integrable". Otherwise you must say "integrable **on** ...".

Comparisons

Let $(f,g) \in PWC([a,+\infty[)^2$ where g is **positive and integrable**.

If $f = O(g)$ at $+\infty$, then f is integrable.

If $|f| \sim g$ at $+\infty$, then f is integrable.

Example : $\int_0^{+\infty} \frac{e^{it}}{\sqrt{\mathrm{ch}(t)}}\mathrm{d}t, \quad \int_2^{+\infty} \ln\left(1+\frac{\sin t}{\sqrt{t}}\right)\mathrm{d}t$

Using a series

Let (b_n) an increasing sequence of $[a,+\infty[$ with $b_0 = a$ and $b_n \longrightarrow +\infty$. We suppose that $\sum_{n\geq 1} \int_{b_{n-1}}^{b_n} f(t)\mathrm{d}t$ converges.

$\int_a^{+\infty} f(t)\mathrm{d}t$ converges and $\int_a^{+\infty} f(t)\mathrm{d}t = \sum_{n=1}^{+\infty} \int_{b_{n-1}}^{b_n} f(t)\mathrm{d}t$ when one of these points is verified :

- f is positive
- $\lim_{n\longrightarrow+\infty} \int_{b_{n-1}}^{b_n} |f(t)|\mathrm{d}t = 0$
- $f(x) \xrightarrow[x\longrightarrow+\infty]{} 0$ and $(b_n - b_{n-1})$ is bounded
- f has real values and has a constant sign on each $[b_{n-1}, b_n]$

Integration by parts : $\int_1^{+\infty} \frac{\sin t}{t^\alpha}\mathrm{d}t$ with $0 < \alpha \leq 1$.

14.2 Generalised integrals on $[a,b[$ or $]a,b]$

Definition

Let $f \in PWC([a,b[,\mathbb{K})$. We say that $\int_a^b f(t)\mathrm{d}t$ converges when $\lim_{x\longrightarrow b} \int_a^x f(t)\mathrm{d}t$ exists in $\mathbb{K}$.

Riemann integrals

Let $a < b$ and $\alpha \in \mathbb{R}$. The integrals $\int_a^b \frac{\mathrm{d}t}{(b-t)^\alpha}$ and $\int_a^b \frac{\mathrm{d}t}{(t-a)^\alpha}$ converge $\Leftrightarrow \alpha < 1$.

Let $f \in PWC([a,b[,\mathbb{K})$. If $\exists \alpha < 1$ such that $\lim\limits_{x\longrightarrow b} (b-x)^\alpha f(x) = 0$, then $\int_a^b f(t)\mathrm{d}t$ converges.

14.3 Generalised integrals on an open interval

Let $-\infty \leq a < b \leq +\infty$

Definition

Let $f \in PWC(]a,b[,\mathbb{K})$. $\int_a^b f(t)\mathrm{d}t$ converges when $\exists c \in]a,b[: \int_a^c f(t)\mathrm{d}t$ and $\int_c^b f(t)\mathrm{d}t$ converge.

In that case, $\int_a^b f(t)\mathrm{d}t = \int_a^c f(t)\mathrm{d}t + \int_c^b f(t)\mathrm{d}t$

Examples : $\Gamma(x) = \int_0^{+\infty} t^{x-1}e^{-t}\mathrm{d}t$, $\int_0^{+\infty} \frac{\mathrm{d}t}{t^\alpha}$, $\int_1^{+\infty} \sin t \ln\left(\frac{t^2+1}{t^2-1}\right)\mathrm{d}t$

14.4 Computing integrals

Let $-\infty \leq a < b \leq +\infty$

Integration by parts

Let $(f,g) \in C^1(]a,b[)^2$. The existence of two of the terms $\int_a^b f'g$, $[fg]_a^b$, $\int_a^b fg'$ implies the existence of the third. In that case, $\int_a^b f'g = [fg]_a^b - \int_a^b fg'$

NEVER USE THIS FORMULA TO PROVE CONVERGENCE, always compute antiderivatives of the form $\int \varphi(t)\mathrm{d}t$. This equation is for computing a convergent integral.

Example : $\int_0^1 \frac{\ln(1-t^2)}{t^2}\mathrm{d}t$

Changing variables

Let $]a,b[$ and $]\alpha,\beta[$ two open intervals, $f \in C^0(]a,b[,\mathbb{K})$ and $\varphi :]\alpha,\beta[\longrightarrow]a,b[$ of class C^1 a strictly increasing bijection.

Then $\int_a^b f(t)\mathrm{d}t$ and $\int_\alpha^\beta f(\varphi(u))\varphi'(u)\mathrm{d}u$ are of the same nature and equal when one converges.

Example : $\displaystyle\int_0^{+\infty} e^{-t^2}\mathrm{d}t = \frac{1}{2}\int_0^{+\infty} \frac{e^{-u}}{\sqrt{u}}\mathrm{d}u$

14.5 Integration of comparison relations

Let $-\infty < a < b \leq +\infty$

Convergent case : comparison of the remainders

Let $(f, \varphi) \in PWC([a, b[, \mathbb{K})^2$ with φ **positive and integrable** on $[a, b[$.

If $f = O(\varphi)$ at the vicinity of b, then f is integrable on $[a, b[$ and $\displaystyle\int_x^b f = O\left(\int_x^b \varphi\right)$ (same for o)

Let $(f, g) \in PWC([a, b[, \mathbb{K})^2$ **both positive** with $f \underset{b}{\sim} g$ and g integrable on $[a, b[$.

Then f is too with $\displaystyle\int_x^b f \sim \int_x^b g$

Example : two-term asymptotical expansion of $\displaystyle\int_0^x \frac{1-\cos t}{t^{5/2}}\mathrm{d}t$, equivalent of $\displaystyle\int_x^{+\infty} e^{-t^2}\mathrm{d}t$

Divergent case : comparison of the partial integrals

Let $(f, \varphi) \in PWC([a, b[, \mathbb{K})^2$ with φ **positive and non-integrable** on $[a, b[$.

If $f = O(\varphi)$ at the vicinity of b, then $\displaystyle\int_a^x f = O\left(\int_a^x \varphi\right)$ (same for o)

Let $(f, g) \in PWC([a, b[, \mathbb{K})^2$ **both positive** with $f \underset{b}{\sim} g$ and g non-integrable on $[a, b[$.

Then f is also non-integrable with $\displaystyle\int_a^x f \sim \int_a^x g$

Example : expasion of $F(x) = \displaystyle\int_x^{+\infty} \frac{e^{-t}}{t}\mathrm{d}t$ when $x \longrightarrow 0^+$

14.6 Parametric Integrals

Let I an interval of $\mathbb{R}$.

Dominated Convergence Theorem

Let (f_n) a sequence of PWC functions on I. If :

- (f_n) converges simply towards $f \in PWC(I)$
- There exists φ **integrable** on I with $\forall n \in \mathbb{N}, \quad \forall t \in I, \quad |f_n(t)| \leq \varphi(t)$

Then the f_n and f are integrable on I with $\displaystyle\int_I f_n(t)\mathrm{d}t \xrightarrow[n\longrightarrow+\infty]{} \int_I f(t)\mathrm{d}t$

Example $I_n = \int_0^{+\infty} \frac{\mathrm{d}t}{1+t^2+t^n e^{-t}}\mathrm{d}t$

Integration term by term

Let $\sum u_n$ a series of functions with each $u_n \in PWC(I)$. If :

- Each u_n is integrable on I
- $\sum u_n$ converges simply on I towars $S \in PWC(I)$
- $\sum \left(\int_I |u_n| \right)$ converges

Then $S = \sum_{n=0}^{+\infty} u_n$ is integrable on I with $\int_I \left(\sum_{n=0}^{+\infty} u_n(t) \right) \mathrm{d}t = \sum_{n=0}^{+\infty} \left(\int_I u_n(t)\mathrm{d}t \right)$

Example $\int_0^1 \frac{\ln t}{t-1}\mathrm{d}t = \sum_{n=1}^{+\infty} \frac{1}{n^2}$

Continuity of a parametric integral

Let X a non-empty part of a finite-dimensional normed vector space E and T an interval of $\mathbb{R}$. Let $f : X \times T \longrightarrow \mathbb{K}$. If :

- $\forall x \in X, \quad t \longmapsto f(x,t)$ is piecewise continuous on T
- $\forall t \in T, \quad x \longmapsto f(x,t)$ is continuous on X
- There exists $\varphi : T \longrightarrow \mathbb{R}$ **integrable** on T so that $\forall (x,t) \in X \times T, \quad |f(x,t)| \leq \varphi(t)$

Then $g : x \longmapsto \int_T f(x,t)\mathrm{d}t$ is defined and continuous on X.

You can replace the third point by "$\forall a \in X$, there exists a vicinity V of a and $\varphi : T \longrightarrow \mathbb{R}$ integrable on T so that $\forall (x,t) \in V \times T, \quad |f(x,t)| \leq \varphi(t)$" (domination at the vicinty of every point).
You can also dominate on all segments of X if $X \subset \mathbb{R}$.

Example : $g(x) = \int_0^{+\infty} e^{-xt}\sqrt{x+t^2}\mathrm{d}t$

Differentiating a parametric integral

Let X, T intervals of $\mathbb{R}$ and $f : X \times T \longrightarrow \mathbb{K}$. If :

- $\forall t \in T, \quad x \longmapsto f(x,t)$ is of class C^1 on X
- $\forall x \in X, \quad t \longmapsto f(x,t)$ is integrable on T
- $\forall x \in X, \quad t \longmapsto \dfrac{\partial f}{\partial x}(x,t)$ is piecewise continuous on T
- For every segment S of X, there exists $\varphi : T \longrightarrow \mathbb{R}$ integrable on T so that :

  $$\forall (x,t) \in S \times T, \quad \left|\frac{\partial f}{\partial x}(x,t)\right| \leq \varphi(t)$$

Then $g : \left\{\begin{array}{ccc} X & \longrightarrow & \mathbb{K} \\ x & \longmapsto & \displaystyle\int_T f(x,t)\mathrm{d}t \end{array}\right.$ is of class C^1 on X with $\forall x \in X, \quad g'(x) = \displaystyle\int_T \frac{\partial f}{\partial x}(x,t)\mathrm{d}t$

Example : $g : x \longmapsto \displaystyle\int_0^{+\infty} \frac{e^{ixt}}{1+t^3}\mathrm{d}t$

Multiple differentiation of a parametric integral

Let X, T intervals of $\mathbb{R}$, $p \in \mathbb{N}^*$ and $f : X \times T \longrightarrow \mathbb{K}$. If :

- $\forall t \in T, \quad x \longmapsto f(x,t)$ is of class C^p on X
- $\forall k \in [\![0, p-1]\!], \quad \forall x \in X, \quad t \longmapsto \dfrac{\partial^k f}{\partial x^k}(x,t)$ is integrable on T
- $\forall x \in X, \quad t \longmapsto \dfrac{\partial^p f}{\partial x^p}(x,t)$ is piecewise continuous on T
- For every segment S of X, there exists $\varphi : T \longrightarrow \mathbb{R}$ integrable on T so that :

  $$\forall (x,t) \in S \times T, \quad \left|\frac{\partial^p f}{\partial x^p}(x,t)\right| \leq \varphi(t)$$

Then $g : \left\{\begin{array}{ccc} X & \longrightarrow & \mathbb{K} \\ x & \longmapsto & \displaystyle\int_T f(x,t)\mathrm{d}t \end{array}\right.$ is of class C^p on X,

with $\forall k \in [\![1, p]\!], \quad \forall x \in X, \quad g^{(k)}(x) = \displaystyle\int_T \frac{\partial^k f}{\partial x^k}(x,t)\mathrm{d}t$

Example : $\Gamma : \left\{\begin{array}{ccc} \mathbb{R}_+^* & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \displaystyle\int_0^{+\infty} t^{x-1}e^{-t}\mathrm{d}t \end{array}\right. \in C^\infty(\mathbb{R}_+^*)$

14.7 TD

Ex 62

Convergence and value of $\displaystyle\int_0^{+\infty} \frac{\ln(t)}{1+t^2}\mathrm{d}t$

Ex 63

Let $f(x) = \displaystyle\int_0^x \ln(\ln(1+t))\mathrm{d}t$.

Prove that f is defined on $]0,+\infty[$ and give an equivalent of $f(x)$ when $x \longrightarrow 0^+$

Ex 64

Let $I = \displaystyle\int_0^1 x^x \mathrm{d}x$

1) Prove the convergence of I.

2) Compute $I_k = \displaystyle\int_0^1 \frac{x^k \ln(x)^k}{k!}\mathrm{d}x$ using the variable change $u = x^{k+1}$.

3) Prove that $I = \displaystyle\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^n}$

Ex 65

Let $f \in C^0([0,1], \mathbb{R}_+^*)$. Consider $G(x) = \displaystyle\int_0^1 \exp(x\ln(f(t)))\mathrm{d}t$,

and let $F(x) = \left(\displaystyle\int_0^1 f(t)^x \mathrm{d}t\right)^{1/x}$. Prove that $F(x) \underset{0}{\sim} \exp\left(\displaystyle\int_0^1 \ln(f(t))\mathrm{d}t\right)$

15 Reduction

15.1 First tools

We consider E a nontrivial $\mathbb{K}$-vector space (not necessarily finite-dimensional).
Reminder : a subsapce F is stable by $u \in L(E)$ when $u(F) \subset F$

Stability

If $u, v \in L(E)$ commute, then $\mathrm{Ker}(v)$ and $\mathrm{Im}(v)$ are stable by u.

Let F a subspace of E stable by $u \in L(E)$. The **induced endomorphism** $u_F \in L(F)$ is defined by $\forall x \in F, \quad u_F(x) = u(x) \in F$.

Cancelling polynomials

$P \in \mathbb{K}[X]$ **cancels** $u \in L(E)$ when $P(u) = 0_{L(E)}$.

The set $CI_u = \{P \in \mathbb{K}[X] | P(u) = 0\}$ is the **cancelling ideal** of u.

If u has a nonzero cancelling polynomial, then there exists a unique unitary polynomial π_u so that $I_u = \pi_u \mathbb{K}[X]$. In this case, all cancelling polynomials of u are multiples of π_u. π_u is the **minimal polynomial** of u.

Example : If E is finite-dimensional, then all endomorphisms have a nonzero cancelling polynomial.

If $u \in L(E)$ has a cancelling polynomial P and a stable subspace F, then $P(u_F) = 0$

Reminder : $u \in L(E)$ is nilpotent if there exists $k \in \mathbb{N}$ so that $u^k = 0$. The smallest k so that $u^k = 0$ is u's nilpotence index.

Nilpotence

Let $u \in L(E)$ nilpotent of index r and $x \notin \mathrm{Ker}(u^{r-1})$. Then $(x, u(x), ..., u^{r-1}(x))$ is independent.

Suppose E finite-dimensional of dimension n. For all nilpotent endomorphisms u, there exists a basis of E in which the matrix of u is strictly upper triangular.
Every nilpotent matrix of $M_n(\mathbb{K})$ is similar to a strictly upper triangular matrix.

Kernel Lemma

Let $(P_1, ..., P_r) \in \mathbb{K}[X]^r$ a family of polynomials.

We suppose that they are **mutually coprime** : $\forall i \neq j, \quad P_i \wedge P_j = 1$.

We define P their product : $\displaystyle\prod_{k=1}^{r} P_k$

Then $\boxed{\forall u \in L(E), \quad \mathrm{Ker} P(u) = \bigoplus_{k=1}^{r} \mathrm{Ker} P_k(u)}$

15.2 Eigenvalues and eigenvectors

Definition

$\lambda \in \mathbb{K}$ is an **eigenvalue** of $u \in L(E)$ is there exists $x \in E \setminus \{0\}$ so that $u(x) = \lambda x$. In that case, x is an **eigenvector** associated to the eigenvalue λ.

The set of the eigenvalues of u is called the **spectrum** of u and written $\mathrm{sp}(u)$.

Let $\lambda \in \mathrm{sp}(u)$. The **eigenspace** associated to λ is the vector space of the associated eigenvectors : $E_\lambda(u) = \mathrm{Ker}(u - \lambda Id)$

The same definitions goes for matrices when E is finite-dimensional.

If u and v commute then the eigenspaces of one are stable by the other.

Properties of eigenspaces

- Let $\lambda_1, ..., \lambda_p$ mutually distinct eigenvalues of u. Then the $E_{\lambda_i}(u)$ are in direct sum.
- Let $F \subset E$ stable by u. Then $E_\lambda(u_F) = E_\lambda(u) \cap F$
- If E is finite-dimensional and $\lambda_1, ..., \lambda_p$ distinct in $\mathrm{sp}(u)$, then $\displaystyle\sum_{i=1}^{p} \dim E_{\lambda_i}(u) \leq \dim E$

Link with cancelling polynomials

- If $x \in E_\lambda(u)$ and $P \in \mathbb{K}[X]$ then $P(u)[x] = P(\lambda)x$
- If P cancels u then all eigenvalues of u are roots of P : $\mathrm{sp}(u) \subset Z(P)$
- If u has a minimal polynomial π_u then $Z(\pi_u) = \mathrm{sp}(u)$

15.3 Characteristic polynomial

We suppose E finite-dimensional.

Definition

Let $A \in M_n(\mathbb{K})$. The **characteristic polynomial** of A is $\chi_A(X) = \det(XI_n - A)$

We have $\chi_A = X^n - \mathrm{Tr}(A)X^{n-1} + ... + (-1)^n \det A$

$\boxed{\forall \lambda \in \mathbb{K}, \quad \lambda \in \mathrm{sp}(A) \Leftrightarrow \chi(\lambda) = 0}$

Let $u \in L(E)$. Since two similar matrices have the same characteristic polynomial, we can define $\chi_u = \chi_A$ where A is the matrix of u in any basis.

Useful properties

- If F is stable by u then χ_{u_F} divides χ_u.
- If F is stable by u and χ_u is totally separated then χ_{u_F} is also totally separated.
- If $\dim E = n$ and $\chi_u = \prod_{i=1}^{n}(X - \lambda_i)$ then $\mathrm{Tr}(u) = \sum_{i=1}^{n} \lambda_i$ and $\det u = \prod_{i=1}^{n} \lambda_i$

Definition

The **order of multiplicity** $m(\lambda)$ of $\lambda \in \mathrm{sp}(u)$ is its order of multiplicity as a root of χ_u.

We have $\forall \lambda \in \mathrm{sp}(u), \quad 1 \leq \dim E_\lambda(u) \leq m(\lambda) \leq \dim E$

Cayley-Hamilton Theorem

$\forall u \in L(E), \quad \chi_u(u) = 0$ and $\forall A \in M_n(\mathbb{K}), \quad \chi_A(A) = 0$.

As a consequence, $\deg(\pi_u) \leq n$

15.4 Diagonalisation

E is still supposed finite-dimensional of dimension n.

Definition

$u \in L(E)$ is said to be **diagonalisable** if there exists a basis of E in which its matrix is diagonal.

$u \in L(E)$ is diagonalisable $\Leftrightarrow$ there exists a basis of E of eigenvectors of u.

$A \in M_n(\mathbb{K})$ is **diagonalisable** when $\exists P \in GL_n(\mathbb{K}), \quad \exists D \in D_n(\mathbb{K}) : A = PDP^{-1}$

Vectorial criteria of diagonalisability

If $\mathrm{sp}(u) = (\lambda_1, ..., \lambda_p)$ where the λ_i are distinct, then the following properties are equivalent :

1) $\displaystyle\bigoplus_{i=1}^{p} E_{\lambda_i}(u) = E$
2) $\displaystyle\sum_{i=1}^{p} \dim E_{\lambda_i}(u) = \dim E$

Using the characteristic polynomial

$u \in L(E)$ is diagonalisable $\Leftrightarrow$ χ_u is totally separated and $\forall \lambda \in \mathrm{sp}(u), \quad \dim E_\lambda(u) = m(\lambda)$

If χ_u is totally separated with simple roots, then u is diagonalisable.

Using cancelling polynomials

There is equivalence between :

1) u is diagonalisable.
2) u has a cancelling polynomial that is totatlly separated and has simple roots.
3) π_u is totally separated and has simple roots.

Let $u \in L(E)$ diagonalisable and $F \subset E$ stable by u. Then u_F is diagonalisable.

15.5 Trigonalisation

E is still finite dimensional of dimension n.

Definition

$u \in Ł(E)$ is **trigonalisable** if there exists a basis of E in which the matrix of u is upper triangular.

$A \in M_n(\mathbb{K})$ is trigonalisable when it is similar to an upper trangular matrix.

Using the characteristic polynomial

$u \in L(E)$ is trigonalisable $\Leftrightarrow$ χ_u is totally separated on $\mathbb{K}$.

When $\mathbb{K} = \mathbb{C}$, all endomorphisms are trigonalisable.

Consequence : Cayley-Hamilton in $\mathbb{C}$.

If u is trigonalisable then if $\mathrm{sp}(u) = (\mu_1, ..., \mu_n)$, $\mathrm{Tr}(u) = \displaystyle\sum_{i=1}^{n} \mu_i, \quad \det(u) = \prod_{i=1}^{n} \mu_i$

Characterisation of nilpotent endomorphisms

The following properties are equivalent :

1) u is nilpotent
2) There exists a basis in which the matrix of u is strictly upper triangular
3) $\chi_u = X^n$
4) u is trigonalisable with $\mathrm{sp}(u) = \{0\}$

Using cancelling polynomials

The following properties are equivalent :

1) u is trigonalisable
2) u has a cancelling polynomial that is totally separated
3) π_u is totally separated

Let $u \in L(E)$ diagonalisable and $F \subset E$ stable by u. Then u_F is diagonalisable.

Fine trigonalisation

If u has a cancelling polynomial that is totally separated, then there exists a basis of E in which the matrix of u is of the form : $\begin{pmatrix} \lambda_1 I_{n_1} + N_1 & & (0) \\ & \ddots & \\ (0) & & \lambda_p I_{n_p} + N_p \end{pmatrix}$ where the N_i are nilpotent.

15.6 TD

Ex 66

Simultaneous reduction. Let E a finite-dimensional vector space of dimension n. We consider $u, v \in L(E)$ that commute ($v \circ u = u \circ v$)

1) We suppose that u and v are diagonalisable. Prove that they are simultaneously diagonalisable : that there exists a basis of E in which the matrices of u **and** v are diagonal.

2) We suppose that u and v are trigonalisable. Prove that they are simultaneously trigonalisable : that there exists a basis of E in which the matrices of u **and** v are upper triangular.

Ex 67

Let $A \in M_n(\mathbb{K})$ and $\varphi : \left\{ \begin{array}{ccc} M_n(\mathbb{K}) & \longrightarrow & M_n(\mathbb{K}) \\ M & \longmapsto & AM \end{array} \right.$

1) Prove that φ is diagonalisable $\Leftrightarrow$ A is diagonalisable.

2) We suppose A diagonalisable. Reduce φ : determine $\mathrm{sp}(\varphi)$ and a basis of $M_n(\mathbb{K})$ in which the matrix of φ is diagonal.

Ex 68

Let u, v two commuting endomorphisms of a finite-dimensional vector space with v nilpotent. Prove that $\det(u + v) = \det(u)$.

16 Complements on Euclidians

Let E an euclidian space with an inner product $(\cdot|\cdot)$. We note $E^* = L(E, \mathbb{R})$.

Riesz's representation lemma

Let $f \in E^*$. Then $\quad \exists! a \in E, \quad \forall x \in E, \quad f(x) = (a|x)$

16.1 Symmetries : the spectral theorem

Definition

$u \in L(E)$ is a **symmetry** when $\forall (x,y) \in E^2, \quad (x|u(y)) = (u(x)|y)$

The subspace of $L(E)$ of the symmetries of E is written $S(E)$.

Characterisation of symmetries

Let B an *orthonormal* basis of E and $u \in L(E)$. $u \in S(E) \Leftrightarrow \mathrm{mat}_B(u)$ is a symmetric matrix.

The subspace $S_n(\mathbb{R})$ of $M_n(\mathbb{R})$ composed of the symmetric matrices is of dimension $\dfrac{n(n+1)}{2}$

Stability

Let $u \in S(E)$. If F is stable by u then $F^\perp$ is also stable by u.

Orthogonal eigenspaces

The eigenspaces of a symmetric endomorphism are orthogonal.

Stable subspaces for endomorphisms of $\mathbb{R}$-vector spaces

We suppose E to be a nontrivial finite-dimensional $\mathbb{R}$-vector space.

Let $u \in L(E)$. There exists a subspace F of E that is stable by u with $\dim F \in \{1, 2\}$

This can be rephrased : "all endomorphisms of an $\mathbb{R}$-vector space have a stable line or a stable plane"

Lemma : spectral theorem in dimension 2

If E is an eucildian of dimension 2, then all symmetries of E are diagonalisable.

Spectral Theorem

Let u a symmetric endomorphism of an euclidian space E. Then :

- E is the orthogonal direct sum of the eigenspaces of u.
- E has an orthonormal basis of eigenvectors of u.
- u is diagonalisable in an orthonormal basis.

In terms of matrices : let $S \in S_n(\mathbb{R})$. Then : $\exists O \in O_n(\mathbb{R}), \quad \exists D \in D_n(\mathbb{R}), \quad S = ODO^T$

We say that "real symmetries are orthogonally diagonalisable".

16.2 Applications of the spectral theorem

Definition

The **spectral radius** of $u \in S(E)$ is $\rho(u) = \max\limits_{\lambda \in \mathrm{sp}(u)} |\lambda|$. We have $|||u||| = \rho(u)$

Positive and positive-definite symmetric matrices

Let $S \in S_n(\mathbb{R})$. S is **positive** when its eigenvalues are positive. We write $S \geq 0$ or $S \in S_n^+(\mathbb{R})$.

S is **positive-definite** when its eigenvalues are stricly positive. We write $S > 0$ or $S \in S_n^{++}(\mathbb{R})$.

$S \geq 0 \Leftrightarrow \forall X \in \mathbb{R}^n, \quad X^T S X \geq 0 \quad$ and $\quad S > 0 \Leftrightarrow \forall X \in \mathbb{R}^n \setminus \{0\}, \quad X^T S X > 0$

The same goes for a symmetric endomorphism. **Example** : let $A \in M_n(\mathbb{R})$. $A^T A, AA^T \geq 0$.

Square root of a positive symmetry

Let $u \in S^+(E)$. $\exists! v \in S^+(E), \quad u = v^2$

For $u \in S^+(E)$ and $A \in S_n^+(\mathbb{R})$, this allows to define $\sqrt{u}$ and $\sqrt{A}$.

Polar Decomposition

Let $M \in GL_n(\mathbb{R})$. $\exists!(O, S) \in O_n(\mathbb{R}) \times S_n^+(\mathbb{R}), \quad M = OS$.

If M is not inversible, there is existence but not unicity.

16.3 Reduction of isometries

Reminder : simplification of $u \in O(\mathbb{R}^2)$

If $\det u = 1$ then it is called a **rotation** and its matrix is in the form $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ in a certain orthonormal basis.

If $\det u = -1$ then it is called a **symmetry** and its matrix is in the form $S_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$ in a certain orthonormal basis.

Lemmae

- The isometries with no eigenvalues of an euclidian plane are the rotations of angle $\theta \not\equiv 0[\pi]$.
- Let $u \in O(E)$ and F stable by u. Then $F^\perp$ is stable by u.

Reduction of isometries

Let $u \in O(E)$. There exists an orthonormal basis of E in which the matrix of u is of the form :

$$\begin{pmatrix} I_p & & & & (0) \\ & -I_q & & & \\ & & R_{\theta_1} & & \\ & & & \ddots & \\ (0) & & & & R_{\theta_r} \end{pmatrix} \quad \text{with } \forall i \in [\![1, r]\!], \quad \theta_i \in \,]-\pi, 0[\, \cup \,]0, \pi[$$

16.4 Vectorial functions

We consider E a finite-dimensional normed $\mathbb{K}$-vector space ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), I an interval of $\mathbb{R}$, $f : I \longrightarrow E$ and $a \in I$.

Definition

f is differentiable at a if its slope $\dfrac{f(t) - f(a)}{t - a}$ has a limit in E when $t \longrightarrow a$.

Like for functions in $\mathbb{R}$, we define functions of class D^k, of class C^k and we have the usual properties on linear combinations and sums.

Let B a bilinear map from $E \times F$ to G, $f : I \longrightarrow E$, $g : I \longrightarrow G$ differentiable at a.
Then $t \longrightarrow B(f(t), g(t))$ is differentiable at a with $B(f, g)'(a) = B(f'(a), g(a)) + B(f(a), g'(a))$.

Example Let $f : I \longrightarrow GL_n(\mathbb{K})$ of class C^1. Then $t \longrightarrow f(t)^{-1}$ is of class C^1.

Mean inequality - general expression

Let $I = [a, b]$, $f : I \longrightarrow E$, $\varphi : I \longrightarrow \mathbb{R}$ so that :

- f and φ are continuous on $[a, b]$
- f and φ are differentiable on $]a, b[$
- $\forall t \in]a, b[, \quad \|f'(t)\| \leq \varphi'(t)$

Then $\|f(b) - f(a)\| \leq \varphi(b) - \varphi(a)$

Integral of a vectorial function on a segment

We suppose E euclidian.

Let $f : [a, b] \longrightarrow E$ piecewise continuous. There are two definitions of $\displaystyle\int_a^b f$:

- Consider $\varphi : \begin{cases} E & \longrightarrow & \mathbb{R} \\ v & \longmapsto & \displaystyle\int_a^b (v|f(t))\mathrm{d}t \end{cases} \in E^*$.

  By RIESZ's lemma, $\exists! I \in E, \quad \varphi = (I|\cdot)$. We define $\displaystyle\int_a^b f(t)\mathrm{d}t = I$

  This gives the formula $\forall v \in E, \quad \displaystyle\int_a^b (v|f) = \left(v \,\middle|\, \int_a^b f\right)$

- Write $\forall t \in [a, b], \quad f(t) = \displaystyle\sum_{i=1}^n f_i(t) e_i$ so $\forall i \in [\![1, n]\!], \quad f_i \in PWC([a, b], \mathbb{R})$.

  We define $\displaystyle\int_a^b f(t)\mathrm{d}t = \sum_{i=1}^n \left(\int_a^b f_i(t)\mathrm{d}t\right) e_i$. This does not depend on the chosen basis (e_i).

All usual results on integrals remain true thanks to the second definition which links vectorial integration and real integration.

16.5 TD

Ex 69

Vectorial Product
Let E an euclidian of dimension n. We consider B a direct orthonormal basis of E.

We note $[\cdot, ..., \cdot] = \det_B(\cdot, ..., \cdot)$.

1) Let $(x_1, ..., x_{n-1}) \in E^{n-1}$. Prove that $\exists! w \in E, \quad \forall y \in E, \quad [x_1, ..., x_{n-1}, y] = (w|y)$.

We note $x_1 \wedge ... \wedge x_{n-1} = w$

2) Give the coordinates of w in the basis B.

3) Prove that $w = 0 \Leftrightarrow (x_1, ..., x_{n-1})$ is dependent.

4) We suppose that $(x_i)_{i=1}^{n-1}$ to be orthonormal. We complete it with x_n into a direct orthonormal basis of E. Prove that $x_1 \wedge ... \wedge x_{n-1} = x_n$.

Ex 70

Legendre's polynomials : We give $\mathbb{R}[X]$ the inner product $(P|Q) = \int_{-1}^{1} PQ$.

1) Prove that there exists $(P_n)_{n\in\mathbb{N}}$ an orthonormal basis of $\mathbb{R}[X]$
with $\forall n \in \mathbb{N}, \quad \deg P_n = n$.

2) Let $n \in \mathbb{N}$. Prove that $\forall Q \in \mathbb{R}_{n-1}[X], \quad \int_{-1}^{1} Q(t)\frac{\mathrm{d}}{\mathrm{d}t}\Big((t^2-1)P_n'(t)\Big)\,\mathrm{d}t = 0$

3) Prove that $\forall t \in \mathbb{R}, \quad (t^2-1)P_n''(t) + 2tP_n'(t) - n(n+1)P_n(t) = 0$

Ex 71

We consider E and euclidian of dimension n and B an orthonormal basis of E.

Let $u \in L(E)$ that **conserves orthogonality** : $(x|y) = 0 \Rightarrow (u(x)|u(y)) = 0$.

Let $A = \mathrm{mat}_B(u)$

1) Prove that $\forall X \in \mathbb{R}^n, \quad \forall Y \in \mathbb{R}^n, \quad X^TY = 0 \Rightarrow X^TA^TAY = 0$

2) Prove that $\forall X \in \mathbb{R}^n, \quad \exists \lambda \in \mathbb{R}, \quad A^TAX = \lambda X$

3) Prove that $\exists \lambda \in \mathbb{R}, \quad \exists O \in O_n(\mathbb{R}), \quad A = \lambda O$

Ex 72

1) Let $A \in S_n^{++}(\mathbb{R}), \quad B \in S_n(\mathbb{R})$.

Prove that $\exists P \in GL_n(\mathbb{R}), \quad \exists D \in D_n(\mathbb{R}), \quad A = P^TP, \quad B = P^TDP$.

2) Let $(A,B) \in S_n^{++}(\mathbb{R})^2$, and $\alpha, \beta \geq 0$ so that $\alpha + \beta = 1$.

Prove that $\det(\alpha A + \beta B) \geq (\det A)^\alpha \times (\det B)^\beta$

3) Let $(A,B) \in S_n^{++}(\mathbb{R})^2$. Prove that $\det(A+B)^{1/n} \geq (\det A)^{1/n} + (\det B)^{1/n}$

17 Differential Equations

17.1 Scalar linear DEs of the n-th order with constant coefficients

Let $n \in \mathbb{N}, \quad (a_0, ..., a_{n-1}) \in \mathbb{C}^n$. We consider the equation $y^{(n)} + a_{n-1}y^{(n-1)} + ... + a_1 y' + a_0 y = 0$.

Its characteristic polynomial is $C(X) = X^n + \sum_{k=0}^{n-1} a_k X^k$, we factorise it in $\mathbb{C}$: $C = \prod_{i=1}^{r} (X - \lambda_i)^{\alpha_i}$

Then all solutions of the equations are of the form $\boxed{y = \sum_{i=1}^{r} P_i e^{\lambda_i t}}$ where each $P_i \in \mathbb{C}_{\alpha_i - 1}[X]$

17.2 First order scalar DEs

Let I an interval of $\mathbb{R}$ and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Resolved and homogenous equations

We consider the equation $(E_0) : y' + ay = 0$ with $a \in C^0(I)$. Let A an antiderivative of a.

The solutions form a vectorial line : they are all of the form $\boxed{y = \lambda e^{-A(t)}}$ with $\lambda \in \mathbb{K}$

For $x_0 \in I, \quad y_0 \in \mathbb{K}$, the Cauchy problem $\begin{cases} y' + ay &= 0 \\ y(x_0) &= y_0 \end{cases}$ has a unique solution $y \in C^1(I)$

Example : $y' = ty$. A solution that has a zero at a point in I will be constant equal to 0.

Inhomogenous equations : variation of the constant

We consider the equation $(E) : y' + ay = b$ where $(a, b) \in C^0(I)^2$.

Let y_0 a nonzero solution of the associated homogenous equation $y' + ay = 0$.

The general method is to look for a particular solution y_p of the form $y_p = \lambda(t)y_0$ with $\lambda \in C^1(I)$

All solutions of (E) are of the form $y = \alpha y_0 + y_p$ with $\alpha \in \mathbb{K}$

For $x_0 \in I, \quad y_0 \in \mathbb{K}$, the Cauchy problem $\begin{cases} y' + ay &= b \\ y(x_0) &= y_0 \end{cases}$ has a unique solution $y \in C^1(I)$

Example $y' - ty = e^t$

Non-resolved equations

We consider the equation $(E) : ay' + by = c$ with $(a, b, c) \in C^0(I)^3$, where a can cancel itself.

The general method is to solve on intervals on which a never takes the value 0, then to proceed by Analysis-Synthesis in order to find solutions on I.

There is no general result on existence or unicity of solutions of Cauchy problems for equations of this form.

Example : $ty' - \alpha y = 0$ with $\alpha \in \mathbb{R}$

17.3 From scalar n-th order to vectorial first order

Consider the equation $y^{(n)} = a_{n-1}y^{(n-1)} + ... + a_0 y$. We define $X = \begin{pmatrix} y \\ \vdots \\ y^{(n-1)} \end{pmatrix} \in C^1(I, \mathbb{K}^n)$.

We want to solve the equation $X' = \begin{pmatrix} y' \\ \vdots \\ a_{n-1}y^{(n-1)} + ... + a_0 y \end{pmatrix}$ which can be put under the form $X' = AX$ with $A \in C^0(I, M_n(\mathbb{K}))$

17.4 First order vectorial DEs with constant coefficients

Let E a normed $\mathbb{K}-$vector space of dimension n, $a \in L(E)$ and $b \in C^0(I, E)$. Let A the matrix of a and $B(t)$ the matrix of $b(t)$ in a basis of E.

We consider the equations :

- $(E) : y' = a.y + b(t), \quad y \in C^1(I, E)$ in terms of matrices : $X' = AX + B(t), \quad X \in C^1(I, \mathbb{K}^n)$.
- $(E_0) : y' = a.y, \quad y \in C^1(I, E)$ in terms of matrices : $X' = AX, \quad X \in C^1(I, \mathbb{K}^n)$
- $(\mathcal{E}_0) : u' = a \circ u, \quad u \in C^1(I, L(E))$ in terms of matrices : $U' = AU, \quad U \in C^1(I, M_n(\mathbb{K}))$

Reminder : by normal convergence, we can define $\exp(f) = \sum_{k=0}^{+\infty} \frac{f^k}{k!}$ for $f \in L(E)$ and $\exp(M) = \sum_{k=0}^{+\infty} \frac{M^k}{k!}$ for $M \in M_n(\mathbb{K})$. By normal convergence of the differentiated series, if f or M are functions of class C^1 then $\exp(f)$ and $\exp(M)$ are too.

Solving $(\mathcal{E}_0)$

Let $t_0 \in I$. $\quad u_0 : \left\{ \begin{array}{ccc} I & \longrightarrow & L(E) \\ t & \longmapsto & \exp((t-t_0)a) \end{array} \right.$ of matrix form : $\boxed{U_0(t) = \exp((t-t_0)A)}$

is a solution of the Cauchy problem $\left\{ \begin{array}{l} u' = a \circ u \\ u(t_0) = \mathrm{Id}_E \end{array} \right.$ matrix form : $U' = AU, U(t_0) = I_n$

Lastly, $\quad \forall t \in I, \quad u(t) \in GL(E)$

Solving (E_0)

All Cauchy problems $\left\{ \begin{array}{l} y' = a.y \\ y(t_0) = y_0 \end{array} \right.$ have an unique solution $\boxed{y(t) = \exp((t-t_0)a).y_0}$

In matrix terms : "$X' = AX$ and $X(t_0) = X_0$" has for unique solution $\boxed{X(t) = \exp((t-t_0)A)X_0}$

The vector space of solutions of (E_0) is isomorphic to E, so it is of dimension n.

Practical resolution of $X' = AX$.

1) When A is diagonalisable

Let $(V_1, ..., V_n)$ a basis of $\mathbb{K}^n$ of eigenvectors of A, and $(\lambda_1, ..., \lambda_n)$ their associated eigenvalues.

The solution space of (E_0) has for basis the functions $Y_k : \left\{ \begin{array}{ccc} I & \longrightarrow & \mathbb{K}^n \\ t & \longmapsto & e^{\lambda_k t} V_k \end{array} \right.$ for $k \in [\![1, n]\!]$

Example : $X' = AX$ for $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

2) When $\mathbb{K} = \mathbb{R}$ and A is diagonalisable in $\mathbb{C}$ but not in $\mathbb{R}$

In this case, the basis of ffunctions (Y_i) will have paired conjugate complex terms $Y, \overline{Y}$.

Since $\text{Span}(Y, \overline{Y}) = \text{Span}(\text{Re}(Y), \text{Im}(Y))$, replacing each pair $(Y, \overline{Y})$ by $(\text{Re}(Y), \text{Im}(Y))$ yields a basis of the solution space.

3) When A is not diagonalisable in $\mathbb{C}$

In this case we trigonalise A : we write $A = PTP^{-1}$ with $P \in GL_n(\mathbb{K})$ and $T \in T_n^+(\mathbb{K})$.

We define $Y = P^{-1}X$: the system becomes $Y' = TY$, which can be solved from bottom to top.

The final solutions are obtained with the equation $X = PY$

It isn't necessary to compute P^{-1}.

Solving (E)

There exists a solution y_1 of (E).

All solutions of (E) are of the form $\boxed{y_1 + \exp(ta).v}$ with $v \in E$.

In matrix form : $\boxed{X_1 + \exp(tA)\Lambda}$ with $\Lambda \in \mathbb{K}^n$

All Cauchy problems $\begin{cases} y' = a.y + b(t) \\ y(t_0) = y_0 \end{cases}$ for $y_0 \in E$ have a unique solution.

A particular solution can be sought under the form $X_1 = \exp(tA)\Lambda(t)$ or under a simple form (polynomials, exponentials,...)

17.5 First order vectorial DEs

There is no general method for expressing solutions, however we have Cauchy's theorem.

Cauchy's theorem

Let $a \in C^0(I, L(E))$, $\quad b \in C^0(I, E)$, $\quad t_0 \in I$ and $y_0 \in E$.

The Cauchy problem $\begin{cases} y' = a(t).y + b(t) \\ y(t_0) = y_0 \end{cases}$ has a unique solution.

Matrix expression : let $A \in C^0(I, M_n(\mathbb{K}))$, $\quad B \in C^0(I, \mathbb{K}^n)$, $\quad t_0 \in I$ and $X_0 \in \mathbb{K}^n$.

The Cauchy problem $\begin{cases} X' = A(t)X + B(t) \\ X(t_0) = X_0 \end{cases}$ has a unique solution.

As a consequence, the map $y \longmapsto y(t_0)$ is an isomorphism from S_0 (solution space of $y' = a(t).y$) to E, with the corresponding result for $X \longmapsto X(t_0)$.

Let X a solution of $X' = A(t)X + B(t)$. If $\exists t_0 \in I, \quad X(t_0) = 0$ then $\forall t \in I, \quad X(t) = 0$

17.6 Structure of the solution space of a first order vectorial DE

We consider the equation $(E) : y' = a(t).y + b(t)$ of matrix analogue $X' = A(t)X + B(t)$.

We also consider $(E_0) : y' = a(t).y$ of matrix analogue $X' = A(t)X$, and S_0 its solution space.

Definition

A **fundamental system** of (E_0) is a basis $(y_1, ..., y_n)$ of S_0.

Wronskian

Let $(y_1, ..., y_n) \in S_0^n$ and B a basis of E.

The **wronskian matrix** of the family (y_i) in the basis B is $W(t) = \text{mat}_B(y_1(t), ..., y_n(t))$.

The **wronskian** of the family (y_i) is $w(t) = \det(W(t))$

There is equivalence between :

1) $(y_1, ..., y_n)$ is a fundamental system
2) $\exists t \in I, \quad w(t) \neq 0$
3) $\forall t \in I, \quad w(t) \neq 0$

Method : to find a particular solution of (E) using (y_i) a fundamental system, one may consider a function of the form $y = \lambda_1(t)y_1(t) + ... + \lambda_n(t)y_n(t)$.

This corresponds to solving $\lambda_1' y_1 + ... + \lambda_n' y_n = b$ or $W(t)\Lambda'(t) = B(t)$.

To do that, decompose b in the basis (y_i), which is the same as computing $W(t)^{-1}B(t)$.

17.7 Scalar DEs of the n-th order

We consider $a_0, ..., a_{n-1}, b$ continuous functions and the equations :

$(E) : y^{(n)} + a_{n-1}(t)y^{n-1} + ... + a_0(t)y = b(t), \quad (E_0) : y^{(n)} + a_{n-1}(t)y^{n-1} + ... + a_0(t)y = 0.$

The previous paragraphs provide the following results :

- The solution space of E_0 is of dimension n. If $(y_1, ..., y_n)$ is a basis of it and y_p a particular solution of E, then all solutions of E are of the form $\lambda_1 y_1 + ... + \lambda_n y_n + y_p$
- All Cauchy problems with an intial condition at $t_0 : \forall i \in [\![0, n-1]\!], \quad y^{(i)}(t_0) = \alpha_i$ has a unique solution.

17.8 Scalar DEs of the second order

We consider the equation $y'' + a(t)y' + b(t)y = c(t)$.

(It can be put under the form $x'' + p(t)x = f(t)$ using a change of function.)

In this case the wronskian of two solutions y_1, y_2 of $y'' + a(t)y' + b(t)y = 0$ is $w = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$.

It verifies the equation $w' + a(t)w = 0$

Methods for finding solutions of the homogenous equation

A possibility is to look for a simple solution (polynomial, t^α, $\cos, \exp \dots$) :
Example : $t^2y'' - 2ty' + 2y = 0$

Another method is to look for a solution under the form $y(t) = \sum_{n=0}^{+\infty} a_n t_n$:

Example : $4ty'' + 2y' - y = 0$

If you have a solution, you can find another using the wronskian method : if you have y_1 and want a y_2, consider their wronskian w (that satisfies $w' + a(t)w = 0$, so you know its value).

Notice that $\boxed{\frac{w}{y_1^2} = \left(\frac{y_2}{y_1}\right)'}$ **Example** : $x'' + \frac{2}{t}x' + x = 0$ using $x_1 = \frac{\sin t}{t}$

Using one solution, you can use the variation of the constant to find another.
Example : $y'' - \tan t y' + 2y = 0$ on $I =]-\frac{\pi}{2}, \frac{\pi}{2}[$

Finding a particular solution of the complete equation

If (y_1, y_2) is a fundamental system, look for a particular solution y with the conditions :

$\begin{cases} \lambda_1 y_1 + \lambda_2 y_2 &= y \\ \lambda_1' y_1 + \lambda_2' y_2 &= 0 \end{cases}$. This yields a system on λ_1', λ_2' : $\begin{cases} \lambda_1' y_1 + \lambda_2' y_2 &= 0 \\ \lambda_1' y_1' + \lambda_2' y_2' &= c \end{cases}$

The determinant of the system is $w(t) \neq 0$, so this method will always provide a particular solution.

Example : $t^2y'' - 2ty' + 2y = t$

17.9 TD

Ex 73

Cauchy's theorem. Let E a finite-dimensional normed vector space, I an interval of $\mathbb{R}$ and $a \in C^0(I, L(E))$.

1) Let $t_0 \in I$ and $(z_n) \in C^0(I, E)^{\mathbb{N}}$ with $\forall t \in I, \quad z_{n+1}(t) = \int_{t_0}^{t} a(u).z_n(u)\mathrm{d}u$.

Prove that $\sum z_n$ converges normally on all segements of I.

2) Let $h \in C^0(I, E)$ and $t_0 \in I$ so that $\forall t \in I, \quad h(t) = \int_{t_0}^{t} a(u).h(u)\mathrm{d}u$. Prove that $h = 0$

3) Let $t_0 \in I, \quad y_0 \in E, \quad b \in C^0(I, E)$. Prove that the Cauchy problem :

$\begin{cases} y' = a(t).y + b(t) \\ y(t_0) = y_0 \end{cases}$ has a unique solution in $D^1(I, E)$

Ex 74

The Sturm-Liouville method. Let $(E_1) : y'' + p_1 y = 0$ and $(E_2) : y'' + p_2 y = 0$ with $\forall t \in \mathbb{R}, \quad p_1(t) \leq p_2(t)$.

1) Let y a nontrivial solution of $y'' + py = 0$. Prove that zeros of y are **isolated** :

that if $y(z) = 0$ then $\exists \eta > 0, \quad \forall t \in [z - \eta, z + \eta] \setminus \{z\}, \quad y(t) \neq 0$.

2) Let y_1 a nontrivial solution of (E_1) and y_2 a solution of (E_2).

Let $t_1 < t_2$ so that $y_1(t_1) = y_1(t_2) = 0$. Prove that $\exists t \in [t_1, t_2], \quad y_2(t) = 0$.

Hint : consider the wronskian of y_1 and y_2.

3) We suppose $p_1 = p_2$ and that (y_1, y_2) is independent.

Let $t_1 < t_2$ two consecutive zeros of y_1. Prove that $\exists! t \in]t_1, t_2[, \quad y_2(t) = 0$

Ex 75

Gronwall's lemma.

Let $a \in C^0(I, \mathbb{R}_+), \quad b \in C^0(I, \mathbb{R}_+), \quad y \in C^1(I, E), \quad t_0 \in I$ and A the antiderivative of a that vanishes at t_0.

We suppose that $\forall t \geq t_0, \quad \|y'(t)\| \leq a(t)\|y(t)\| + b(t)$.

1) Let $\forall t \in I, \quad F(t) = \int_{t_0}^{t} (b(s) + a(s)\|y(s)\|)\mathrm{d}s$ and $z_0 = \|y(t_0)\|$.

Prove that $\forall t \geq t_0, \quad \|y(t)\| \leq z_0 + F(t)$

2) Dominate $\left(Fe^{-A}\right)'$ in order to prove that :

$$\forall t \geq t_0, \quad \|y(t)\| \leq e^{A(t)}\|y(t_0)\| + e^{A(t)} \int_{t_0}^{t} b(s)e^{-A(s)}\mathrm{d}s$$

PART II

Physics

1 Analyse Dimensionnelle

1.1 Définitions

L'Analyse dimensionnelle est l'étude des dimensions des grandeurs physiques (longueur, masse,...). Elle permet de mieux comprendre les formules et surtout de vérifier si elles sont justes !

Dimensional Analysis is the study of the dimension of physical quantities (length, mass,...). It allows a better understanding of formulas, and it is especially used to verify if they are correct !

Liste des dimensions du Système International (SI)			
Dimension	Symbole	Unité	Symbole
Masse	M	kilogramme	kg
Temps	T	seconde	s
Longueur	L	mètre	m
Température	Θ	kelvin	K
Intensité électrique	I	ampère	A
Quantité de matière	N	mole	mol
Intensité lumineuse	J	candela	cd

Une quantité peut ne pas avoir de dimension (dite "homogène à rien" ou "scalaire") : par exemple $\frac{d}{L}$ où d et L sont des longeurs n'est homogène à rien. De même, des nombres, des pourcentages, des probabilités ... ne sont homogènes à rien.

A quantity can have no dimension (it is said to be "homogenous to nothing" or "scalar") : for instance $\frac{d}{L}$ where d and L are lengths is homogenous to nothing. Similarly, numbers, percentages, probabilities ... are homogenous to nothing.

Critères d'homogénéité d'une formule/equation : *Criteria for the homogeneity of a formula/equation* :

- Les deux côtés d'une égalité ont même dimension. *Both sides of an equation have the same dimension.*
- Deux grandeurs additionnées ensemble ont même dimension. *Two quantities that are added together must have the same dimension.*
- Une expression dans une fonction (log, exp, sin, cos,...) ne doit pas avoir de dimension. *Inside a function (log, exp, sin, cos,...), an expression must have no dimension.*

Attention à utiliser les mêmes unités pour des valeurs de même dimension ! *Be careful to use the same units for homogenous quantities*
Dans une intégrale ou une expression différentiée, les "d_ " comptent ! Par exemple, $[\mathrm{d}t] \equiv T$. *In an integral or a differentiated expression, the "d_" s count ! For example,* $[\mathrm{d}t] \equiv T$
Une formule non homogène est TOUJOURS fausse. *A non-homogenous formula is ALWAYS false.*

Notation : [quantité] ≡ symbole de dimension ou symbole d'unité
Exemple : Soit f une force. $[f] \equiv MLT^{-2}$ ou $[F] \equiv kg.m.s^{-2}$.

1.2 Unités classiques

Symbole	Nom	Description	Equivalents	SI
Hz	Hertz	fréquence	s^{-1}	s^{-1}
rad	radian	angle	1	1
N	Newton	force	$kg.m.s^{-2}$	$kg.m.s^{-2}$
Pa	Pascal	pression	$N.m^{-2}$	$kg.m^{-1}s^{-2}$
J	Joule	énergie	$N.m, C.V, W.s$	$kg.m^2.s^{-2}$
W	Watt	puissance	$J.s^{-1}$, $V.A$	$kg.m^2.s^{-3}$
C	Coulomb	charge électrique	$A.s$, $F.V$	$A.s$
V	Volt	tension électrique	W/A, J/C	$kg.m^2.s^{-3}.A^{-1}$
Ω	Ohm	résistance électrique	V/A	$kg.m^2.s^{-3}.A^{-2}$
F	Farad	capacité d'un condensateur	$A.s.V^{-1}$	$m^{-2}.kg^{-1}.s^4.A^2$
H	Henry	inductance	$V.s.A^{-1}$	$kg.m^2.s^{-2}.A^{-2}$
T	Tesla	intensité d'un champ magnétique	$V.s.m^{-2}$	$kg.s^{-2}.A^{-1}$

1.3 Exercices

Ex 1

On rappelle l'expression de l'intensité de la force de gravitation :$F = G\dfrac{m_1 m_2}{d^2}$
Donner la dimension et l'unité de G.

Ex 2

Sachant que pour une spire de rayon R parcourue par un courant I, le champ magnétique s'écrit : $B = \dfrac{\mu_0 I}{2R}$, donner la dimension et l'unité de μ_0

Ex 3

Sachant que ε_0 est en $F.m^{-1}$, laquelle de ces formules paraît possible ?
a) $\dfrac{\varepsilon_0}{\mu_0} = c^2$
b) $\varepsilon_0 \mu_0 c^2 = 1$
c) $\varepsilon_0 \mu_0 = c$

Ex 4

L'expression de la période T d'oscillations d'un pendule de longeur l tenant une masse m ne dépend que de l, m et g, l'accélération de pesanteur.
Déterminer T par analyse dimensionnelle.

Ex 5

La fréquence f de vibration d'une goutte peut s'écrire sous la forme $f = kR^\alpha \rho^\beta \tau^\gamma$, où :
- k est une constante sans dimension
- R est le rayon de la goutte
- ρ sa masse volumique
- τ est la tension superficielle (force par unité de longueur)
Déterminer α, β, γ.

Constantes physiques

Vitesse de la lumière	c	$= 2,99792458 \,.\, 10^{8}$ m.s^{-1}
Charge élémentaire	e	$= 1,60219 \,.\, 10^{-19}$ C
Nombre d'Avogadro	$\mathcal{N}_{\mathrm{A}}$	$= 6,02204 \,.\, 10^{23}$ mol^{-1}
Constante gravitationnelle	G	$= 6,672 \,.\, 10^{-11}$ N.m^{2}.kg^{-2}
Constante des gaz parfaits	R	$= 8,3144$ J.K^{-1}.mol^{-1}
Constante de Faraday	$\mathcal{F}$	$= 96\,484$ C.mol^{-1}
Constante de Boltzmann	k_{B}	$= 1,38066 \,.\, 10^{-23}$ J.K^{-1}
Constante de Planck	h	$= 6,62617 \,.\, 10^{-34}$ J.s
Masse de l'électron	m_{e}	$= 9,10953 \,.\, 10^{-31}$ kg
Masse du neutron	m_{n}	$= 1,675 \,.\, 10^{-27}$ kg
Masse du proton	m_{p}	$= 1,673 \,.\, 10^{-27}$ kg
Permittivité du vide	ε_0	$= 8,85419 \,.\, 10^{-12}$ F.m^{-1}
Perméabilité du vide	μ_0	$= 4\,\pi \,.\, 10^{-7}$ H.m^{-1}

Masse du Soleil	$1,9891 \,.\, 10^{30}$ kg
Masse de la Terre	$5,9736 \,.\, 10^{24}$ kg
Masse de la Lune	$7,34 \,.\, 10^{22}$ kg
Rayon du Soleil	696 000 km
Rayon de la Terre (équateur)	6 378, 14 km
Rayon de la Lune (équateur)	3 474, 6 km
Distance Soleil-Terre (demi grand axe)	149 597 870 km
Distance Terre-Lune (demi grand axe)	384 400 km

2 Complements on Mechanics

2.1 Reminders

Fundamental Principle of Dynamics (PFD)

For a resulting force $\overrightarrow{f}$ we have $\overrightarrow{f} = \dfrac{\mathrm{d}\overrightarrow{p}}{\mathrm{d}t} = m\overrightarrow{a}$ (for an object of constant mass m).

Théorème du Moment cinétique (TMC)
Angular Momentum Theorem

Here O is the origin of the coordinate system.

We define the "moment cinétique" (*angular momentum*) of an object as $\overrightarrow{L_O} = \overrightarrow{OM} \wedge m\overrightarrow{v}$.

We define the "moment" of a force $\overrightarrow{f}$ as $\overrightarrow{\mathcal{M}}_O(\overrightarrow{f}) = \overrightarrow{OM} \wedge \overrightarrow{f}$.

The TMC is $\boxed{\dfrac{\mathrm{d}\overrightarrow{L_O}}{\mathrm{d}t} = \sum_i \overrightarrow{\mathcal{M}}_O(\overrightarrow{f_i})}$

For a solid, we have $L_O = J\dot{\theta}$ where J is the "moment d'inertie" (*moment of inertia*) of the solid.

Ex 6 Prove the pendulum equation with the TMC.

Lois de Coulomb
Coulomb*'s law*

For an interface with a static friction coefficient f_s and a dynamic friction coefficient f_d, we have :

1) If there is no sliding : $\|\overrightarrow{T}\| \leq f_s\|\overrightarrow{N}$.

2) If there is sliding : $\|\overrightarrow{T}\| = f_d\|\overrightarrow{N}\|$.

With $\overrightarrow{T}$ the tangential reaction force and $\overrightarrow{N}$ the normal (perpendicular) reaction force. In practice we often consider $f = f_s = f_d$ the friction coefficient (approximation).

Ex 7 At what angle α does a slope need to be for a box to slide down it ? (friction coefficient f).

2.2 Energetics

Definition

La puissance d'une force (*The power of a force*) $\overrightarrow{f}$ is $P(\overrightarrow{f}) = \overrightarrow{f}.\overrightarrow{v}$

Le travail élémentaire (*The elemental work*) $\delta W(\overrightarrow{f}) = P(\overrightarrow{f})\mathrm{d}t = \overrightarrow{f}.\mathrm{d}\overrightarrow{OM}$, pour un déplacement élémentaire (*For an elemental movement*) $\mathrm{d}\overrightarrow{OM}$.

Le travail selon un arc (*The work along an arc*) (AB) is $W(\overrightarrow{f})_{A\to B} = \displaystyle\int_{M\in(AB)} \delta W(\overrightarrow{f})$ (intégrale curviligne dépendant du chemin suivi/*Curved integral that depends on the chosen path*).

Théorème de l'Energie cinétique
Kinetic Energy Theorem

We define "l'énergie cinétique" (*kinetic energy*) of an object of mass m and speed $\overrightarrow{v}$ by : $E_c = \frac{1}{2}mv^2$.

Loi de la puissance mécanique (*The Mechanical Power law*) : $\dfrac{\mathrm{d}E_c}{\mathrm{d}t} = \displaystyle\sum_i P(\overrightarrow{f_i})$

Loi de l'énergie cinétique (*The Kinetic Energy Law*) : $E_c(t_B) - E_c(t_A) = \displaystyle\sum_i W_{A\to B}(\overrightarrow{f_i})$.

Ex 8

A skier of mass m travels a distance d and descends a height h on a straight line. He glides without friction (frottement) and has a starting speed v_0. Calculate his speed at the end of the slope.

Definition

A force $\overrightarrow{f}$ is **conservative** if its "travail" (*work*) does not depend on the path it takes.

We can write $W_{A\to B} = E_p(A) - E_p(B)$ where E_p is a function of the position called "**énergie potentielle**" (*potential energy*).

We thus have $\delta W(\overrightarrow{f}) = -\mathrm{d}E_p$ so $\overrightarrow{f}.\mathrm{d}\overrightarrow{OM} = \delta W(\overrightarrow{f}) = -\mathrm{d}E_p$

Finally, $\overrightarrow{f}$ is said conservative when $\overrightarrow{f} = -\overrightarrow{\mathrm{grad}}E_p$. $\overrightarrow{f}$ is said to derive of E_p.

Examples

- Poids (*weight*) $\overrightarrow{P} = -mg\overrightarrow{u_z}$
- Force gravitationnelle (*grativational force*) $\overrightarrow{F} = -G\dfrac{m_1m_2}{r^2}\overrightarrow{u_r}$
- Force de rappel élastique (*elastic force*) $\overrightarrow{f} = -k(l - l_0)\overrightarrow{u}_{\text{spring}\to\text{system}}$
- Force d'inertie d'entraînement (*centrifugal force*) $\overrightarrow{f_{ie}} = m\omega^2\overrightarrow{HM}$

Energie Mécanique
Mechanical Energy

We define $E_m = E_p + E_c$.

For a system subject only to conservative forces, we have the conservation of the Mechanical Energy : $\frac{\mathrm{d}E_m}{\mathrm{d}t} = 0$.

In general, with a sum of the non conservative forces $\overrightarrow{f_{NC}}$, we have the **Mechanical Energy Theorem** : $\Delta E_m = W(\overrightarrow{f_{NC}})$.-

Energétique du solide indéformable en rotation
Energetics of a non-deformable rotating solid

The kinetic energy of a point of the solid M_i is : $E_c(M_i) = \frac{1}{2}m_i v_i^2 = \frac{1}{2}m_i r_i^2 \dot{\theta}^2$.

The total kinetic energy is : $E_c = \sum_i E_c(M_i) = \sum_i \frac{1}{2}m_i r_i^2 \dot{\theta}^2 = \frac{1}{2}\left(\sum_i m_i r_i^2\right)\dot{\theta}^2 = \frac{1}{2}J\dot{\theta}^2$.

The "puissance" (*power*) of a force : $\overrightarrow{f}$ is $P(\overrightarrow{f}) = \mathcal{M}(\overrightarrow{f})\dot{\theta}$

The Kinetic Energy Theorem : $\frac{\mathrm{d}E_c}{\mathrm{d}t} = \sum_i P(\overrightarrow{f_i})$

2.3 Equilibria

We consider that $\overrightarrow{F} = F(x)\overrightarrow{u_x}$ derives of the potential energy $E_p(x)$.

By conservation of the mechanical energy, $E_m = E_c + E_p = \text{cte}$.

But $E_c \geq 0$ so $\boxed{E_p \leq E_m}$.

To know the movement around x, we use $\overrightarrow{F} = -\frac{\mathrm{d}E_p}{\mathrm{d}x}\overrightarrow{u_x}$.

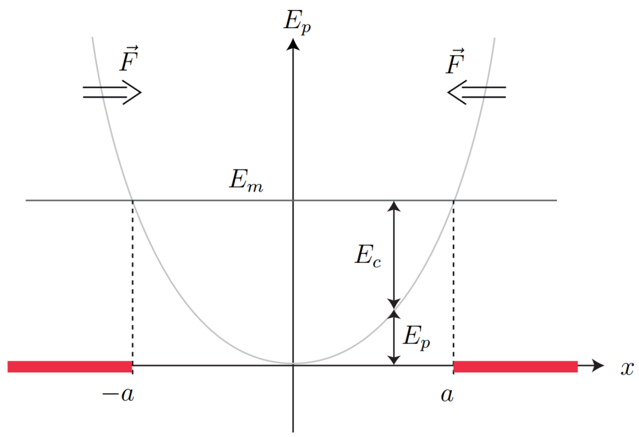
Positions d'équilibre
Equilibrium Positions

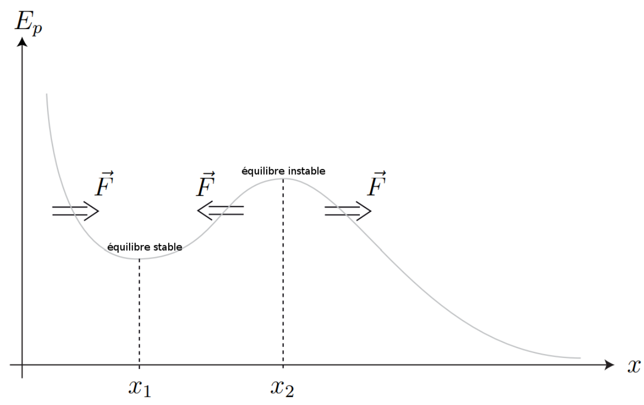
To have an equilibrium at x_{eq} there needs to be $\overrightarrow{F}(x_{eq}) = \overrightarrow{0}$.

Because $\overrightarrow{F} = -\frac{\mathrm{d}E_p}{\mathrm{d}x}\overrightarrow{u_x}$, we have an equilibrium at $x_{eq} \Leftrightarrow \frac{\mathrm{d}E_p}{\mathrm{d}x}(x_{eq}) = 0$

An equilibrium point is said **stable** if the object stays there even if its movement is slightly perturbed.

An equilibrium point is said **unstable** if the object doesn't stay there if its movement is slightly perturbed.





Criteria for the type of stability

Suppose that x_{eq} is an equilibrium point.

If $\dfrac{\mathrm{d}^2 E_p}{\mathrm{d}x^2}(x_{eq}) > 0$ then x_{eq} is stable.

If $\dfrac{\mathrm{d}^2 E_p}{\mathrm{d}x^2}(x_{eq}) < 0$ then x_{eq} is unstable.

Proof

Let $\varepsilon = x - x_{eq}$. By the PFD :

$$m\ddot{\varepsilon} = F(x_{eq} + \varepsilon) \approx F(x_{eq}) + \varepsilon \frac{\mathrm{d}F}{\mathrm{d}x}(x_{eq}) = 0 - \varepsilon \frac{\mathrm{d}^2 E_p}{\mathrm{d}x^2}(x_{eq})$$

- If $\dfrac{\mathrm{d}^2 E_p}{\mathrm{d}x^2}(x_{eq}) > 0$, then we define $\omega_0 = \sqrt{\dfrac{1}{m}\dfrac{\mathrm{d}^2 E_p}{\mathrm{d}x^2}(x_{eq})}$ and we have $\ddot{\varepsilon} + \omega_0^2 \varepsilon = 0$: harmonic oscillations around x_{eq}, a stable positon.

  We have $\varepsilon(t) = A\cos(\omega_0 t + \varphi)$ with A, φ two constants.

- If $\dfrac{\mathrm{d}^2 E_p}{\mathrm{d}x^2}(x_{eq}) < 0$, then we define $p = \sqrt{-\dfrac{1}{m}\dfrac{\mathrm{d}^2 E_p}{\mathrm{d}x^2}(x_{eq})}$ and we have $\ddot{\varepsilon} - p^2 \varepsilon = 0$: exponential divergence, an unstable position.

  We have $\varepsilon(t) = Ae^{pt} + Be^{-pt}$ with A, B two constants.

2.4 Exercises

Ex 9 We consider a pendulum of mass m and length L that can turn at 360 degrees. Study its equilibrium angles.

Ex 10 At what condition on the starting speed v_0 of a cyclist can he go through a loop of radius r ?

Ex 11 A curve turns around an axis at a constant rotation speed ω. We place a ring on the curve so it glides along it. What equation must the curve have in order to have an equilibrium point at every point on the curve ?

2.5 Movement in a central force field

Definition

A "force centrale" (*central force*) $\overrightarrow{f}$ is a force of the form $\overrightarrow{f} = f(r)\overrightarrow{u_r}$ (spherical coordinates).

An object subject to a central force always moves along the same place. We therefore use polar coordinates to describe its movement.

Proof

By the "Théorème du Moment cinétique", $\dfrac{\mathrm{d}\overrightarrow{L}}{\mathrm{d}t} = \overrightarrow{\mathcal{M}}(\overrightarrow{f})$. Yet $\mathcal{M}(\overrightarrow{f}) = \overrightarrow{OM} \wedge \overrightarrow{f} = r\overrightarrow{u_r} \wedge f(r)\overrightarrow{u_r} = \overrightarrow{0}$

So $\overrightarrow{L}$ is constant, so $\overrightarrow{OM}$, which is always perpendicular to $\overrightarrow{L}$, is always in the same plane.

We now use polar coordinates.

Constante des aires
The area constant

The quantity $C = r^2\dot{\theta}$ is constant for an object subject to only $\overrightarrow{f}$.

Indeed, $\|\overrightarrow{L}\|$ is constant and is equals to $r \times m\|\overrightarrow{v_\theta}\| = mr^2\dot{\theta}$.

Definition

A "force Newtonienne" (*Newtonian force*) is a central force of the form $\overrightarrow{f} = -\dfrac{K}{r^2}\overrightarrow{u_r}$.

It derives from the potential energy $E_p(r) = -\dfrac{K}{r}$

We now suppose $\overrightarrow{f}$ Newtonian.

Lois de Képler
Kepler*'s laws*

For the gravitation force $\overrightarrow{f} = -G\dfrac{mM}{r^2}\overrightarrow{u_r}$ (M is the mass of the attracting body), we have :
1) Loi des orbites/Orbits Law All objects have an elliptical trajectory with O as one of the foyers.
2) Loi des aires/Area Law The area covered by the vector $\overrightarrow{OM}$ during two time intervals of equal lengths is equal.
3) Loi des périodes/Period Law For a trajectory of period T and of semi-major axis a (for a circle, that's the radius), we have $\dfrac{T^2}{a^3} = cte$.

Proof for a circular trajectory :

On $\overrightarrow{u_r}$, with the PFD $-m\dfrac{v^2}{r} = -G\dfrac{mM}{r^2}$ so $v = \sqrt{\dfrac{GM}{r}}$. Then $\dfrac{T^2}{r^3} = \dfrac{\left(\dfrac{2\pi r}{v}\right)^2}{r^3} = \dfrac{4\pi^2}{GM}$ is a constant.

Energie potentielle effective
Effective potential energy

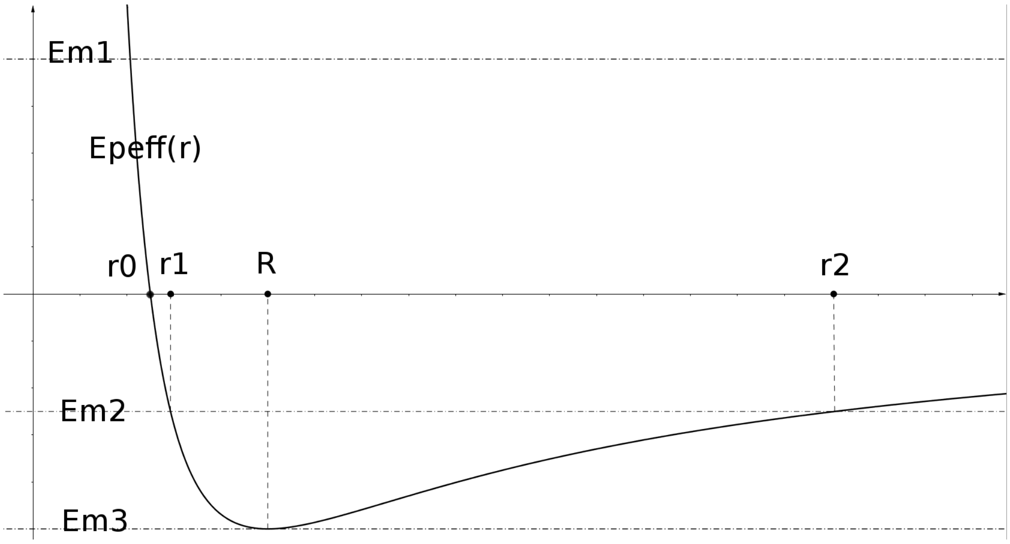
We have $E_m = E_p + E_c = -\frac{K}{r} + \frac{1}{2}mv^2$

Yet $\overrightarrow{v} = \dot{r}\overrightarrow{u_r} + r\dot{\theta}\overrightarrow{u_\theta}$, so $v^2 = \dot{r}^2 + r^2\dot{\theta}^2$.

Finally $E_m = -\frac{K}{r} + \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) = -\frac{K}{r} + \frac{1}{2}mr^2\dot{\theta}^2 + \frac{1}{2}m\dot{r}^2$

We define the **Energie potentielle effective** (*Effective potential energy*) $\boxed{E_{peff} = -\frac{K}{r} + \frac{1}{2}mr^2\dot{\theta}^2 = E_p(r) + \frac{1}{2}m\frac{C^2}{r^2}}$ (with $C = r^2\dot{\theta}$) the area constant.

We always have $E_{peff} \leq E_m$, and we can use E_{peff} to describe the movement, similarly to a usual potential energy.



- For E_{m2}, to satisfy $E_{peff}(r) \leq E_{m2}$, necessarily you need $r \in [r_1, r_2]$. The orbit is an ellipsis : the radius oscillates between r_1 and r_2.
- For E_{m3}, to satisfy $E_{peff}(r) \leq E_{m3}$, necessarily you need $r = R$. The orbit is a circle of radius R.
- For E_{m1}, all $r \geq r_0$ allow $E_{peff}(r) \leq E_{m1}$. The object can leave to infinity.

Circular Trajectories
Now we suppose $r = cte$.

Particular Values for a circular orbit

We have $\boxed{v = \sqrt{\frac{GM}{r}}}$ (because by the PFD on $\overrightarrow{u_r}$, $-m\frac{v^2}{r} = -G\frac{mM}{r^2}$)

We have $E_p = -G\frac{mM}{r}$, $\quad E_c = \frac{1}{2}mv^2 = \frac{1}{2}G\frac{mM}{r}$ $\quad$ so $\boxed{E_m = -\frac{1}{2}G\frac{mM}{r}}$

Remark : We admit that for an ellipsis of semi-major axis a, $E_m = -\frac{GMm}{2a}$.

Première vitesse cosmique
First cosmic speed

The first cosmic speed v_1 is the speed of a satellite at a circular orbit that is at ground level ($r = R_T$).

We have $v_1 = \sqrt{\frac{GM_T}{R_T}} \approx 8km/s$

Seconde vitesse cosmique
Second cosmic speed

The second cosmic speed, or liberation speed, is the speed required to leave the Earth's pull and leave to space.

We consider that such a satellite leaves towards infinity with a final speed close to zero.

The conservation of the mechanical energy gives : $E_m(\text{start}) = E_m(\text{inifnity})$, therefore :

$\frac{1}{2}mv_2^2 - G\frac{mM_T}{R_T} = \frac{1}{2}mv_\infty^2 - G\frac{mM_T}{r_\infty}$.
We consider $v_\infty = 0$ (since v_2 is the minimal liberation speed), and $r_\infty = +\infty$ (since the object approaches infinity).

Finally $E_m = 0 = \frac{1}{2}mv_2^2 - G\frac{mM_T}{R_T}$ so $v_2 = \sqrt{\frac{2GM_T}{R_T}} = \sqrt{2}v_1 \approx 11km/s$

Ex 12

For a black hole of mass M, there exists a distance R from its center of mass past which not even light can escape. Compute R.

Ex 13

A satellite describes an ellipsis such that at its closest point to Earth (perigee) it is $d_p = 200km$ away from the Earth and so that at its farthest point (apogee), it is $d_a = 5.9 \times 10^3 km$ away from the Earth.

Draw its trajectory, compute the mechnical energy E_m of the satellite and its revolution period T.

We give the speed of the satellite at its apogee : $v_a = 3.5 \times 10^2 ms^{-1}$. Determine its speed at the perigee. Explain why it is faster at its perigee than at its apogee.

2.6 Homework Correction

2.6.1 Correction of Ex 7

If light can't escape at radius R it means that the escape velocity at R is the speed of light c. Therefore we have $c = \sqrt{\frac{2GM}{R}}$ so $\boxed{R = \frac{2GM}{c^2}}$.

2.6.2 Correction of Ex 8

The semi-major axis of the ellipsis is $a = \frac{d_p + d_a}{2}$, therefore the mechanical energy of the satellite is $\boxed{E_m = -\frac{1}{2}G\frac{mM}{a}}$

To calculate the period we use KEPLER's law : $\dfrac{T^2}{a^3} = \dfrac{4\pi^2}{GM_T}$, so $\boxed{T = \sqrt{a^3\dfrac{4\pi^2}{GM_T}}}$.

By conservation of the angular momentum (central force), we have $d_a v_a = d_p v_p$ so $\boxed{v_p = \dfrac{d_a v_a}{d_p}}$. It is faster because we have $C = r^2\dot\theta$ therefore the closer the satellite the faster its rotation speed $\dot\theta$.

3 More Mechanics

3.1 Changing Referentials

In this lesson $\mathcal{R}$ will be a galilean referential of origin O and $\mathcal{R}'$ will be the moving referential of origin O'.

For a translation

$\overrightarrow{v}_{M/\mathcal{R}} = \overrightarrow{v}_{M/\mathcal{R}'} + \overrightarrow{v}_{O'/\mathcal{R}}$.
$\overrightarrow{a}_{M/\mathcal{R}} = \overrightarrow{a}_{M/\mathcal{R}'} + \overrightarrow{a}_{O'/\mathcal{R}}$.

For a constant rotation around a fixed axis

We note $\overrightarrow{\Omega}_{\mathcal{R}'/\mathcal{R}}$ the angular velocity vector (*vecteur instantané de rotation*) between $\mathcal{R}'$ and $\mathcal{R}$, and H the projection of M on the rotation axis.

$\overrightarrow{v}_{M/\mathcal{R}} = \overrightarrow{v}_{m/\mathcal{R}'} + \overrightarrow{\Omega}_{\mathcal{R}'/\mathcal{R}} \wedge \overrightarrow{HM}$
$\overrightarrow{a}_{M/\mathcal{R}} = \overrightarrow{a}_{M/\mathcal{R}'} - \Omega^2_{\mathcal{R}'/\mathcal{R}}\overrightarrow{HM} + 2\overrightarrow{\Omega}_{\mathcal{R}'/\mathcal{R}} \wedge \overrightarrow{v}_{M/\mathcal{R}'}$

In all cases, $\overrightarrow{v}_{M/\mathcal{R}} = \overrightarrow{v}_{M/\mathcal{R}'} + \overrightarrow{v_d}$, where $\overrightarrow{v_d}$ is the **drive speed** (*vitesse d'entraînement*) of the referential. $\overrightarrow{v_d}$ is the speed of the fixed point on the moving referential that coincides with M at the analysed instant. The acceleration of that point is called the **driving acceleration** $\overrightarrow{a}_d$

For acceleration we also have the CORIOLIS acceleration $\overrightarrow{a_c}$ depending on the type of movement of $\mathcal{R}'$. In every case, $\overrightarrow{a}_{M/\mathrm{R}} = \overrightarrow{a}_{M/\mathrm{R}'} + \overrightarrow{a_d} + \overrightarrow{a_c}$.

- For a translation : $\overrightarrow{a_d} = \overrightarrow{a}_{O'/\mathrm{R}}$, and $\overrightarrow{a_c} = \overrightarrow{0}$
- For a rotation : $\overrightarrow{a_d} = -\Omega^2_{\mathcal{R}'/\mathcal{R}}\overrightarrow{HM}$ and $\overrightarrow{a_c} = 2\overrightarrow{\Omega}_{\mathcal{R}'/\mathcal{R}} \wedge \overrightarrow{v}_{M/\mathcal{R}'}$

Fundamental Principle of Dynamics in $\mathcal{R}'$

We have the PFD in $\mathcal{R}$: $m\overrightarrow{a}_{M/\mathcal{R}} = \overrightarrow{F}$, so $m\overrightarrow{a}_{M/\mathcal{R}'} + m\overrightarrow{a_d} + m\overrightarrow{a_c} = \overrightarrow{F}$
So $m\overrightarrow{a}_{M/\mathcal{R}'} = \overrightarrow{F} - m\overrightarrow{a_d} - m\overrightarrow{a_c}$

We define the **drive force** (or centrifugal force for a rotation) $\overrightarrow{f}_d = -m\overrightarrow{a_d}$ and the CORIOLIS force $\overrightarrow{f}_{cor} = -m\overrightarrow{a_c}$.

The PFD in $\mathcal{R}'$ thus writes $m\overrightarrow{a}_{M/\mathcal{R}'} = \overrightarrow{F} + \overrightarrow{f}_d + \overrightarrow{f}_{cor}$.

If $\mathcal{R}'$ moves in a translation then $\overrightarrow{f}_d = -m\overrightarrow{a}_{O'/\mathcal{R}}$, and $\overrightarrow{f}_{cor} = \overrightarrow{0}$

If $\mathcal{R}'$ moves in a rotation then $\overrightarrow{f}_{cen} = m\Omega^2_{\mathcal{R}'/\mathcal{R}}\overrightarrow{HM}$, and $\overrightarrow{f}_{cor} = -2m\overrightarrow{\Omega}_{\mathcal{R}'/\mathcal{R}} \wedge \overrightarrow{v}_{M/\mathcal{R}'}$

Ex 14

Describe the movement of a ball sliding without friction with a starting speed $\overrightarrow{v_0}$ on a roundabout rotating at ω (constant). Do the description in both referentials.

Angular Momentum Theorem in $\mathcal{R}'$

$$\left.\frac{\mathrm{d}\overrightarrow{L}_{M/\mathcal{R}'}}{\mathrm{d}t}\right)_{\mathcal{R}'} = \sum_i \overrightarrow{\mathcal{M}_i} + \overrightarrow{\mathcal{M}}(\overrightarrow{f}_d) + \overrightarrow{\mathcal{M}}(\overrightarrow{f}_{cor})$$

Ex 15

A point A is moving along the horizontal so that $x_A(t) = X\cos(\omega t)$. We have attached a pendulum of length L and mass m to A. Calculate the movement equation of the pendulum.

3.2 Examples

3.2.1 Driving force

The definition of $\overrightarrow{g}$ is changed on the Earth to account for the rotation of the Earth around its axis : $\overrightarrow{g} = \overrightarrow{A_{grav}} - \overrightarrow{a_d}$. This makes a very little difference however.

3.2.2 Cyclones

If a gas particle wants to reach a depression, it is deviated by the Coriolis force.

3.3 Static of Fluids

Equation of the static of fluids

$\boxed{\overrightarrow{\mathrm{grad}}P = \overrightarrow{f}_v}$

Where $\overrightarrow{f}_v$ is the resultant of the volumic forces (its dimension is [Force]/[Volume])

Reminder : the elementary force $\delta\overrightarrow{F}$ applied by the pressure P on an elementary surface δS of normal $\overrightarrow{n}$ is $\delta\overrightarrow{F} = -P\delta S\overrightarrow{n}$.

Ex 16

My watch can sustain $P = 10$ bar. At what depth can I dive ?

Ex 17

The height of the mercury in a mercury thermometer in ordinary conditions is P_0 is $h = 76cm$. What is the volumic mass of mercury ?

Archimides's force

For an immobile object of volume V submerged in a fluid of volumic mass ρ, we have by the PFD $\overrightarrow{\Pi} + \rho V\overrightarrow{g} = \overrightarrow{0}$, where $\overrightarrow{\Pi}$ is Archimides's force that "pushes" the object up.

Therefore, $\overrightarrow{\Pi} = -\rho V\overrightarrow{g}$

Ex 18

What proportion of the volume of an iceberg is submerged under water ?

4 Ideal Gases and Statistic Physics

4.1 The Kinetic Pressure Model

The average of a quantity X is written $\langle X \rangle$ or $\overline{X}$.

Expression the pressure in function of the average quadratic speed

We define the pressure P of a fluid on a surface the physical quantity that satisfies $\mathrm{d}\overrightarrow{F} = -P\mathrm{d}\overrightarrow{S}$.

For a gaz, we have $\boxed{P = \dfrac{n^* m^* u^2}{3}}$, where m^* is the mass of a gas molecule, $n^* = \frac{N}{V}$ (the molecular density) and u is the average quadratic speed $u = \sqrt{\langle v^2 \rangle}$.

Definition

We can give a kinetic defintion of the temperature : $\boxed{\langle E_c \rangle = \frac{3}{2} k_B T}$

This gives the Ideal Gases relation : $\boxed{PV = nRT}$

An **Ideal Gas** is a model of the behaviour of any gas when $P \longrightarrow 0$. This is when the gas particules don't interact with each other.

The Van der Waals equation

We consider that the internal energy of a gas molecule is $E = \dfrac{\alpha}{r^{12}} - \dfrac{\beta}{r^6}$.

The Ideal Gas equation is perfected by Van der Waals with $\boxed{(P + \dfrac{an^2}{V^2})(V - nb) = nRT}$.

b is the molar volume of the molecules and a a constant that depends on the gas.

The isothermic atmosphere model

The application of the static of fluids gives $\boxed{P(z) = P_0 \exp\left(-\dfrac{m_p g z}{k_B T}\right)}$

The Boltzmann factor

Let ε be an attainable energy state for a particle. The number of particules at the state ε is :

$$\boxed{N_\varepsilon = \frac{N_{tot}}{Z} \exp\left(-\frac{\varepsilon}{k_B T}\right)}$$

$\exp\left(-\dfrac{\varepsilon}{k_B T}\right)$ is called the Boltzmann factor. Z is a constant that allows $\sum\limits_{\varepsilon} N_\varepsilon = N$.

This has a probabilistic writing : $\boxed{p_\varepsilon = \dfrac{1}{Z} \exp\left(-\dfrac{\varepsilon}{k_B T}\right)}$

For convenience we define $\beta = \frac{1}{k_B T}$.

Ex 19

We consider a particle of magnetic moment $\overrightarrow{M}$ in a magnetic field $\overrightarrow{B}$. We give the potential energy attached to its interaction with the field : $E_p = -\overrightarrow{M}.\overrightarrow{B}$.

It has two possible states : "spin $\frac{1}{2}$" : $\overrightarrow{M}$ and $\overrightarrow{B}$ are positively colinear, and "spin $-\frac{1}{2}$: they are anticolinear.

Calculate the populations of the two spins.

Thermal Capacity at constant volume

We define $\boxed{C_V = \frac{\partial \overline{E}}{\partial T}}$

There are two different models we can use in physics. There is the **classic model** that says that the energy is a continuous function (it can have an infinite amount of values), and there is the **quantum model** that says that energy levels are "quantified" : they are discrete, finite.

The two following theorems belong to the classic approach.

The Maxwell-Boltzmann Law

For an ideal gas composed of N particles, the probability to mesure a particle with a speed $\overrightarrow{v_{mes}}$ within the intervals :

$\overrightarrow{v_{mes}}.\overrightarrow{u_x} \in [v_x, v_x + \mathrm{d}v_x]$, $\overrightarrow{v_{mes}}.\overrightarrow{u_y} \in [v_y, v_y + \mathrm{d}v_y]$, $\overrightarrow{v_{mes}}.\overrightarrow{u_z} \in [v_z, v_z + \mathrm{d}v_z]$

Is $\boxed{\mathrm{d}p_{v_x,v_y,v_z} = A^{-1} e^{-\frac{\beta m v^2}{2}} \mathrm{d}v_x \mathrm{d}v_y \mathrm{d}v_z}$

A is a normalisation constant : $A = \int e^{-\frac{\beta m v^2}{2}} \mathrm{d}v_x \mathrm{d}v_y \mathrm{d}v_z$

Equipartition Theorem

If in the classic limit the energy can be written under the form $E = aX^2 + b$ where X is either a variable of position x_i or a variable of impulsion p_i, and where a and b do not depend on X,

The **Equipartition Theorem** gives $\boxed{\langle aX^2 \rangle = \frac{1}{2} k_B T}$

For example for one particle of gas, $E = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m} + E_p(x, y, z)$

By the Equipartion Theorem, $\left\langle \frac{p_{x,y,z}^2}{2m} \right\rangle = \frac{1}{2} k_B T$, so $\langle E_c \rangle = \frac{3}{2} k_B T$

Capacity of an Ideal Gas

For a **monoatomic ideal gas**, $E = \frac{p_{1,x}^2}{2m} + \frac{p_{1,y}^2}{2m} + \frac{p_{1,z}^2}{2m} + ... + \frac{p_{N,z}^2}{2m} + E_p$

$E_p = 0$ because the gas is ideal. We therefore have by the Equipartition Theorem :

$\langle E \rangle = \frac{3 N k_B T}{2} = \frac{3}{2} nRT$ Finally $\boxed{C_v = \frac{3}{2} nR}$

For a **Diatomic ideal gas**, there are two extra degrees of liberty for each particle (the rotation of that particle comparing to its partner), so we have $\boxed{C_v = \frac{5}{2} nR}$

5 Thermodynamics

5.1 Models and Definitions

Intensive/Extensive parameters

Let $\Sigma = \Sigma_1 \cup \Sigma_2$ be a system.
A quantity X is said to be **intensive** when $X(\Sigma) = X(\Sigma_1) = X(\Sigma_2)$.
For example : P, T, ρ are intensive.
A quantity X is said to be **extensive** when $X(\Sigma) = X(\Sigma_1) + X(\Sigma_2)$.
For example V, m, H, S, U, C_v are extensive.

Notation : Let X be an extensive quantity. X_m is the molar associated quantity and x is the massic associated quantity.

Definition

A **stationnary** state is a state where the parameters no longer change.
An **equilibrium** state is a state where the parameters do not change and were there are no exchanges between the system and its exterior.
You have **local thermic equilibrium** when the temperature and the pressure are defined at every point (not necessarily uniform).

Ideal gases

At a microscopic scale : the molecules' size and interactions are neglected.
At a macroscopic scale : at a thermodynamic equilibrium, $PV = nRT$.

JOULE's law : for an ideal gas, $U = U(T)$, $H = H(T)$.

Partial pressures : $P = \sum_i P_i$

Internal energy and enthalpy

The **internal energy** U is so that $E_{tot} = U + E_{\text{macro}}$

We define the **enthalpy** H by : $H = U + PV$

Heat capacity at constant volume $C_v = \left.\dfrac{\partial U}{\partial T}\right)_{V=cte}$

Heat capacity at constant pressure $C_p = \left.\dfrac{\partial H}{\partial T}\right)_{P=cte}$

Heat capacity of an ideal condensed phase

For a condensed phase $U = U(T) = H = H(T)$. We have $C_v = C_p = C$

And $\boxed{\mathrm{d}U = C\mathrm{d}T,\ \mathrm{d}H = C\mathrm{d}T}$

Joule's relations and Mayer's relations for ideal gases

By JOULE's law, $U = U(T)$ and $H = H(T)$ for an ideal gas so $\boxed{\mathrm{d}U = C_v \mathrm{d}T \text{ and } \mathrm{d}H = C_p \mathrm{d}T}$

For an ideal gas we define $\gamma = \dfrac{C_p}{C_v}$.

We have MAYER's relations : $\boxed{C_{v,m} = \dfrac{R}{\gamma - 1} \text{ and } C_{p,m} = \dfrac{\gamma R}{\gamma - 1}}$

Work of the pressure forces

We have $\delta W = -P_{ext}\mathrm{d}V$, so $\boxed{W = -\int P_{ext}\mathrm{d}V}$

Transformation types

- An **isochoric** transformation has $V = cte$
- An **isobaric** transformation has $P = cte$
- A **monobaric** transformation has $P_{ext} = cte$
- An **isothermic** transformation has $T = cte$
- A **monothermic** transformation has $T_{ext} = cte$
- An **adiabatic** transformation has no thermal interactions with the exterior $Q_{ext} = 0$
- A **reversible** transformation can be done the other way

5.2 Thermodynamic Principles

First principle of Thermodynamics

- U is extensive.
- For a transofrmation of a **closed** system, $\boxed{\Delta U + \Delta E_{\text{macro}} = W + Q}$

W is the **work** and Q the **heat** (homogenous to energy)

U is a state function : ΔU does not depend on the transformation.

If the system is at macroscopic rest, $\Delta E_{\text{macro}} = 0$

For a **monobaric transformation**, $\Delta H = W_u + Q \quad (W_u = W - W_{\text{pressure}}$ the **useful work**)

Laplace's law

For a mechanically reversible adiabatic transformation of ideal gases :
$\boxed{PV^\gamma = cte,\ TV^{\gamma-1} = cte \text{ and } T^\gamma P^{1-\gamma} = cte}$

Second principle of Thermodynamics

There exists an extensive quantity S called "entropy" so that for a transformation of a closed system,
$\boxed{\Delta S = S_e + S_c}$.

S_e is the **exchanged entropy** satisfying $S_e = \sum_i \frac{Q_i}{T_i}$ (for heat exchanges Q_i with thermostats at temperatures T_i)

S_c is the **created entropy** satisfying $S_c \geq 0$ in general, with $S_c = 0$ when the transformation is reversible.

S is a state function : ΔS does not depend on the transformation.

Thermodynamic identities

1) : $\boxed{\mathrm{d}U = T\mathrm{d}S - P\mathrm{d}V}$ *2)* : $\boxed{\mathrm{d}H = T\mathrm{d}S + V\mathrm{d}P}$

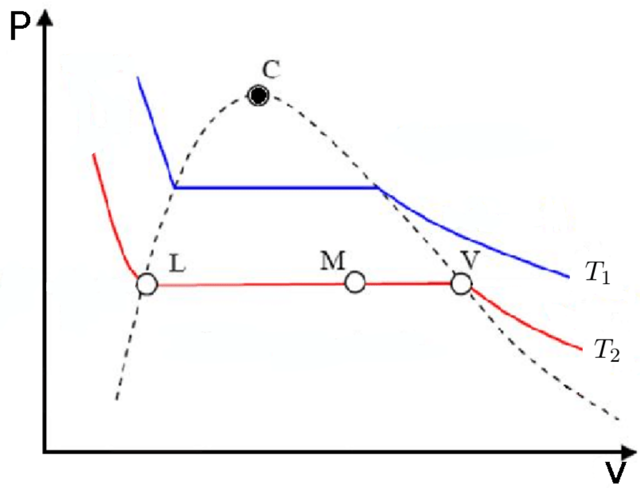
Application : for a perfect gas, $\boxed{\Delta S = \frac{nR}{\gamma - 1}\ln\left(\frac{P_F V_F^\gamma}{P_I V_I^\gamma}\right)}$

For an ideal compressed phase, $\boxed{\Delta S = C\ln\left(\frac{T_F}{T_I}\right)}$

Ex 20 Compute the entropy for an isothermic and reversible dilatation of an ideal gaz from a volume $\frac{V}{2}$ to a volume V.

Ex 21 We have two incompressible bodies of masses m_1 and m_2 initially at rest at temperatures T_1 and T_2. They have massic thermic capacities c_1 and c_2. They exchange heat by conduction until they are at a common temperature T_f. Determine T_f and the created entropy. Discuss the case $m_1 = m_2$ and $c_1 = c_2$.

5.3 Clapeyron diagrams and state transformations



Diphased systems

We have for a liquid/gas diphased system and for all extensive quantities A : $A = x_v A_v + x_l A_l$ with x_v the fraction of gas and x_l the fraction of liquid.

On CLAPEYRON's graph, $\boxed{x_v = \frac{v - v_l}{v_v - v_l} = \frac{MV}{ML}}$

We define the state change enthalpy between state 1 and state 2 : $L_{1\to 2} = H(2) - H(1)$.

The entropy of state change from 1 to 2 is $\Delta_{1\to 2} S = \frac{\Delta_{1\to 2} H}{T}$.

5.4 Industrial Principles

First and Second Principles for a flowing fluid (industrial principles)

For a fluid flowing at a constant massic rate R_m we have :

$\boxed{[e_c + e_{p,ext} + h]_I^O = w_u + q}$, $\boxed{R_m[e_c + e_{p,ext} + h]_I^O = P_u + P_{th}}$ and $\boxed{s_O - s_I = s_e + s_c}$

Ex 22 (JOULE-KELVIN decompression) A perfect gas goes through a pipe and through a porous obstacle so that its pressure goes from P_I to P_O ($P_I > P_O$ and its temperature from T_I to T_O. Show that $h_I = h_O$ and find the massic created entropy s_c.

Ex 23 A perfect gas goes through a nozzle (tuyère) so that its temperature goes from T_I to T_O, its pressure from P_I to P_O and its speed from c_I to c_O. Find the variation in speed.

5.5 Thermic Machines

We consider a fluid going in a cycle. It receives a work W (if $W > 0$ then the machine receives external energy to function (example : fridge), if $W < 0$ then it creates work (example : motor). It is in contact with two thermostats T_w (warm) and T_c (cold) and receives heat from them : Q_w and Q_c. For example if $Q_w > 0$ then the machine receives heat from the warm source.
Over a cycle, $\Delta U = 0$, $\Delta H = 0$ and $\Delta S = 0$.

Carnot-Clausius inequality

We have $0 = \Delta S = S_e + S_c$ so $0 \le S_c = -\frac{Q_w}{T_w} - \frac{Q_c}{T_c}$ so $\boxed{\frac{Q_w}{T_w} + \frac{Q_c}{T_c} \le 0}$

5.5.1 Motors

For a motor, $W < 0, Q_w > 0, Q_c < 0$: the energy is drawn from the warm source in order to produce work.

We define its **yield** $\boxed{\eta = \frac{|\text{interesting}|}{|\text{costly}|} = -\frac{W}{Q_w}}$: always smaller than the reversible yield $\boxed{\eta_{max} = 1 - \frac{T_c}{T_w}}$

5.5.2 Fridges

For a fridge $W > 0, Q_w < 0, Q_c > 0$: it receives work in order to make the cold source colder and the warm source warmer.

We define its efficiency $\boxed{e = \frac{|\text{interesting}|}{|\text{costly}|} = \frac{Q_c}{W}}$: smaller than the reversible efficiency $\boxed{e_{max} = \frac{T_c}{T_w - T_c}}$.

5.5.3 Thermic Pumps

A thermic pump works the same as a fridge, but the objective is to heat up the warm source.

Its efficiency is defined by $\boxed{e = \frac{|\text{interesting}|}{|\text{costly}|} = -\frac{Q_w}{W}}$: smaller than the reversible efficiency $\boxed{e_{max} = \frac{T_w}{T_w - T_c}}$

5.6 TD

Ex 24

To create artificial snow you pulverise water droplets at $T_1 = 10°C$ in air at $T_a = -15°C$. We'll consider the drops to be spherical of radius $R = 0,2mm$, of volumic mass $\rho = 10^3 kg.m^{-3}$ and of massic thermic capacity $c = 4,18.10^3 J.kg^{-1}$.

1) First the drop cools while staying liquid. It receives a thermic transfer $Q = h(T_a - T(t))$ (T is the drop's temperature and $h = 65 W.m^{-2}K^{-1}$). Apply the first principle to a drop between t and $t + \mathrm{d}t$ in order to determine $T(t)$.

2) When the drop reaches $-5°C$, it starts to freeze (so its temperature becomes $0°C$). Find the fraction x of liquid that still has to freeze considering the reaction to be quick and adiabatic. We give $L_{fusion} = 335.10^3 J.kg^{-1}$.

3) How long does it take the drop to solidify ?
Hint : To find the duration τ find an equation on $m(t)$

Ex 25

We consider one mole of ideal gas ($\gamma = 1,4$) undergoing the following cycle :

- Isothermic decompression from $P_A = 2bar$ to $P_B = 1bar$ in contact with a thermostat $T_T = 300K$
- Isobaric evolution to $V_C = 20,5L$ (still in contact with T_T)
- Adiabatic and reversible compression back to state A.

1) Represent the cycle on a (P, V) graph. Is the cycle creating or receiving work ?

2) Find the total entropy variation between A and B then find S_e and S_c. Prove that $A \to B$ is reversible.

3) Find the temperature in C, the work W_{BC}, the heat Q_{BC} as well as S_e and S_c between B and C. Is this cycle possible ?

Ex 26

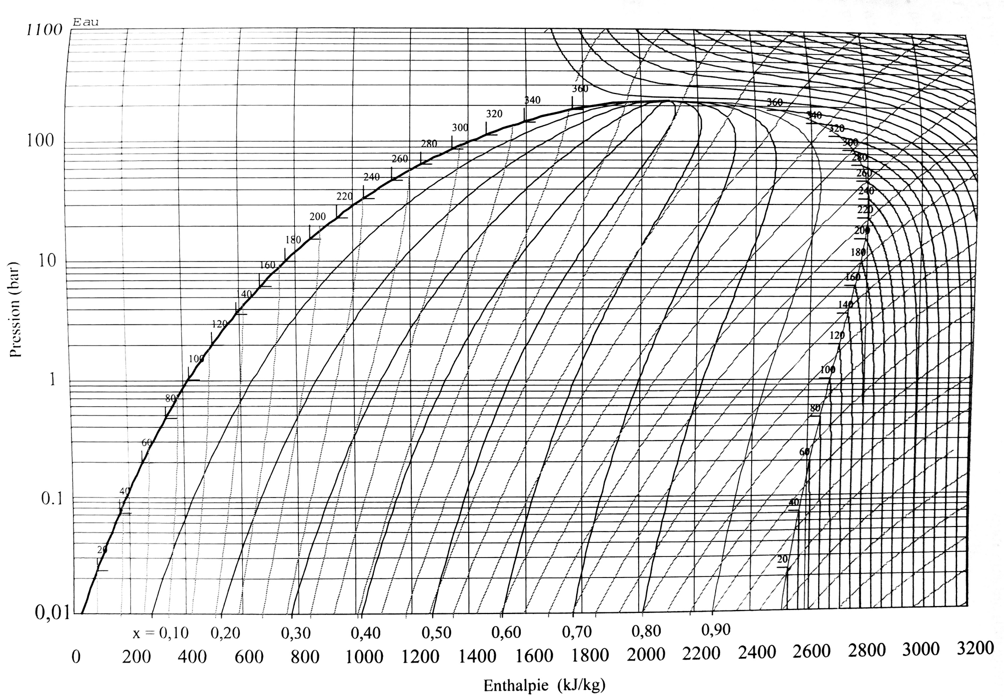
We consider then Carnot cycle for thermic motors that use water :

- State A : liquid water at $P_1 = 0,2bar$, $T_1 = 60°C$.
- $A \to B$: adiabatic and isentropic compression to $P_2 = 15bar$.
- $B \to C$: isobaric heating to $T_2 = 200°C$ so that $P_2 = P_{sat}(T_2)$.
- $C \to D$: total vaporisation. (isobaric and isothermic)
- $D \to E$: adiabatic and isentropic decompression into a liquid-vapour mix at T_1.
- $E \to A$: total condensation.

1) Draw the cycle on the given (P, h) graph.

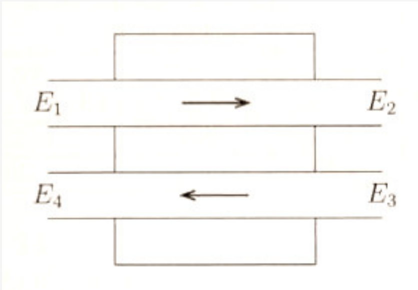
2) Compute all the heat transfers using the values of h on the graph.

3) Compute the yield of the cycle and compare it to the Carnot yield. Explain the causes of irreversibility.



Ex 27

We condider a thermic exchanger (TE) with two circuits of air (considered an ideal gas of molar mass $M = 29g.mol^{-1}$ and of constant $\gamma = 1,4$). It is an open system with two entries with the same mass transfer rates. We suppose that the TE is in permanent regime, that it is heat-isolated and that its functions reversibly.



On one pipe, the air goes from state E_1 (temperature T_1) to state E_2 (at T_2), and at the bottom from E_3, T_3 to $E_4, T4$.

1) Apply the two industrial principles in order to get two relations on the T_i.

2) Prove that $T_1 = T_2$ and $T_3 = T_4$.

3) In reality, $T_1 = 350K$, $T_2 = 290K$, $T_3 = 280K$ and $T_4 = 340K$. Compute numerically the created entropy for a mass $m = 1kg$ of air going through the TE. Comment upon it.

4) In reality the system isn't perfectly isolated : it transfers heat with the atmosphere (Thermostat at T_0). Find the expression of the heat transfer Q received by the air for a mass $m = 1kg$ and give the expression of the created entropy.

6 Optics

6.1 Diopters

Definition

For a transparent medium in which the light has a speed v we define its **index** $n = \dfrac{c}{v}$.
We note the wavelength of a light ray λ_0 in vacuum and λ in other mediums.

A **light ray** is a model of a photon's movement. We represent it by a line.

Fundemental laws of geometric optics

- In a transparent, homogenous and isotropic medium, the light rays are straight lines.
- All light rays are independent.
- A path taken by a light ray can be taken in both directions (**inverse path principle**)

Definition

A **diopter** is an interface between two mediums of different indexes.
We define the **incidence angle** i as the oriented angle bewteen the normal to the diopter's surface and the light ray.

Snell's laws

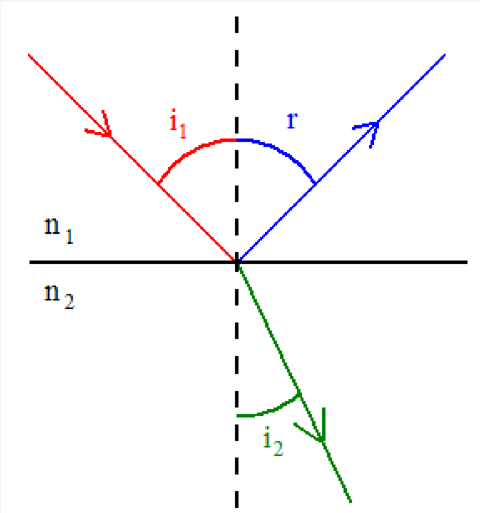
Consider a light ray approching a diopter with an incidence angle i_1 at the interface of two mediums of indexes n_1 and n_2.

First Law : There is a **reflected ray** that returns in the first medium with an angle i'. There is a **refracted ray** that goes through the diopter into the second medium within the incidence plane at an angle i_2.

Second Law : $\boxed{i' = -i \text{ and } n_1 \sin i_1 = n_2 \sin i_2}$

If $n_1 < n_2$, the medium 1 is said to be **less refringent** than the medium 2 and the refracted ray approches the normal to the diopter ($i_2 < i_1$)

If $n_1 > n_2$, the medium 1 is said to be **more refringent** than the medium 2 and the refracted ray goes away from the normal to the diopter ($i_1 > i_2$)



Example : 2 diopters.

Limit refraction

Consider a dioptre at the interface of two mediums with $n_1 < n_2$. There is **limit refraction** when $i_1 \longrightarrow \frac{\pi}{2}$. In that case there is a limit refracted angle α so that $\sin \alpha = \frac{n_1}{n_2}$.

Total reflection

Consider a diopter at the interface of two mediums with $n_1 > n_2$. There is **total reflexion** when $i_2 \longrightarrow \frac{\pi}{2}$. This is reached for a limit incident angle α' so that $\sin \alpha' = \frac{n_2}{n_1}$. If $i_1 > \alpha'$ then there is no refraction.

Dispersive mediums

A medium is said to be **dispersive** when its index depends on the wavelength of the light ray.

Cauchy's empiric model says that for all mediums $\boxed{n = A + \frac{B}{\lambda^2}}$ with $A, B > 0$.

A medium can have an inhomogenous index : salty water, hot air.

Classic exercise : The prism laws.

6.2 Objects and Images

Definition

An **object point** is the intersection of the light rays that go towards an optic system.

An **image point** is the intersection of the light rays that exit an optic system.

Stigmatism

There is **stigmatism** when the image of a point is a point.

For example the mirror is rigorously stigmatic. Furthermore for a mirror $\overline{HA} + \overline{HA'} = 0$

Definition

If the optics systems have a common axis of symmetry, we call it the **optic axis**. The system is then called a **central system**. This will be the case in the majority of our studies.

Aplanetism

There is **aplanatism** when the image of an object that is perpendicular to the optic axis is perpendicular to the optic axis.

A mirror is rigourously aplanetic.

Gauss's conditions

GAUSS's conditions for a central system is to have little inclination on the rays and a small distance between the rays and the optic axis. For a diopter it is to have a small incidence angle.

Under these conditions, we have approximate stigmatism and aplanetism.

Example : the plane diopter.

6.3 Focus points

Definition

An object is said to be **at infinity** when it is infinitely far away from the optic system. The light rays coming from it are parallel (example : the sun's light rays).

Principal focuses

The **object focus** F of an optical system is so that all rays passing by F emerge parallel to the optic axis.

The **image focus** F' of an optical system is so that all rays that enter the system parallely to the optic axis pass by F'.

F and F' always belong to the optic axis.

Secondary focuses

Consider an object at infinity that is not on the optic axis. The rays coming from it are parellel. The emerging rays will all pass by a **secondary image focus** F'_s. All F'_s are in the same plane as F'.

The image of an object by the system is at infinity when its entering rays pass by a **secondary object focus** F_s. All F_s are in the same plane as F.

6.4 Lenses

Types of lenses

There are two types of lenses : **convergent** lenses and **divergent** lenses.

The object focus point of a convergent lens is on its left and its image focus point is on its right. It is the contrary for a divergent lens.

The **focal** of a lens f' is defined by $f' = \overline{OF'}$. It is positive for a converent lens and negative for a divergent one.

A ray passing by the center O of a lens is not deviated.

Lens formulas

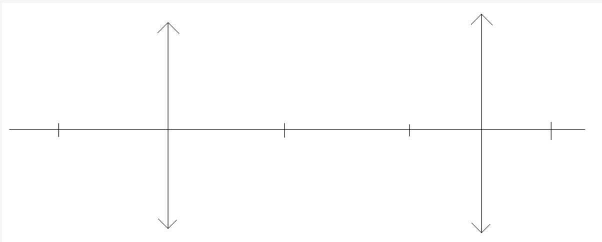
The **enlargement formula** $\boxed{\gamma = \frac{\overline{A'B'}}{\overline{AB}} = \frac{\overline{F'A'}}{\overline{F'O}} = \frac{\overline{FO}}{\overline{FA}}}$

Newton's formula : $\boxed{\overline{FA} \times \overline{F'A'} = \overline{FO} \times \overline{F'O} = -(f')^2}$

Descartes's formula : $\boxed{\frac{1}{\overline{OA'}} - \frac{1}{\overline{OA}} = \frac{1}{f'}}$

6.5 TD

Ex 28



We consider two convergent lenses of focals f'_1 and f'_2 and of centers O_1, O_2 separated by a distance e so that $\overline{F'_1F_2} = e$

1) Complete the diagram

2) Complete the path of a ray passing through the center of the first lens at an angle α.

3 Considering both lens to be an optic system of focal points F and F', compute $\overline{O_2F'}$ and $\overline{O_1F}$

4) Show that if $e = 0$, the system is equivalent to a convergent lens. Give its focal.

Ex 29

Silbermann's method for finding the focal of a convergent lens

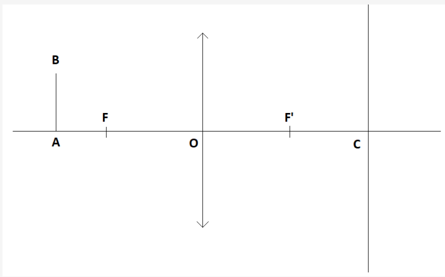
Consider a convergent lens of which we want to know the focal. There is an object AB (with A on the optical axis and B perpendicular to it) on the left of the lens and a screen E on its right. We can adjust the position of the lens and of the screen.

There exists one position such that the image of AB, $A'B'$, is clear on the screen and of the same size but is flipped upside down.

1) Draw this position. *2)* Compute the focal f' in function of $D = AA'$.

Ex 30

Bessel's method : Consider a convergent lens and an object AB on its left. We have a fixed screen E such that $D = \overline{AC}$ is a constant (C is the intersection between the screen and the optical axis). The only movable part is the lens.



Show that if $D > 4f'$ then there exists two distances $x_{1,2} = \overline{AO}$ in order to have a clear image of AB on the screen and find their values when $D > 4f'$. Let $d = |x_1 - x_2|$. Compute f' depending on D and d.

Ex 31

Galileo's telescope. We consider two lenses : one convergent L_1 of focal $f_1' = 60cm$ and one divergent L_2 one its right of focal $f_2' = -5cm$.

1) On what condition is the system **afocal**? (This means that the image of an object at infinity is at infinity.)

2) Draw the system in that situation. Draw the path of a ray going through O_1 at a nonzero angle with the optical axis. Use the secondary object focus method.

3) We note B the used secondary object focus point used in *2)* and α' the angle between (OB_2) and the optical axis. Compute the enlargement $G = \dfrac{\alpha'}{\alpha}$ using the values of the focals.

7 Ondulatory Optics

7.1 Definitions and theorems

Definition

We consider a **scalar model** of light rays : we modelise the signal by a scalar $s(M,t)$.

A **spherical wave** is of the from $\underline{s}(M,t) = \dfrac{A}{r}e^{i\omega t - kr}$
A **plane wave** is of the form $\underline{s}(M,t) = s_0 e^{i\omega t - kz}$

We define $\varphi(M,t) = \text{Arg}(\underline{s})$ the **phase**. We have $\boxed{\dfrac{\varphi(A,t) - \varphi(B,t)}{2\pi} = \dfrac{AB}{\lambda}}$

The **path difference** $\boxed{\delta = \lambda\dfrac{\Delta\varphi}{2\pi}}$. The **order of interference** : $p = \dfrac{\delta}{\lambda}$

Interferences are constructive when $\delta = k\lambda$ and destructive when $\delta = \frac{2k+1}{2}\lambda$.

A **wave surface** is a surface for which all the waves have the same phase and went through the same optical path.

The **intensity** of a light ray is $I = \langle s^2 \rangle$

Malus's Theorem

After an arbitrary number of refractions and reflexions, the light rays are always perpendicular to the wave surfaces.

Fresnel's interference formula

Consider two light sources S_1 and S_2 of intensities I_1 and I_2. The combined intensity is I.

- If the two sources have the same pulsation and originate from the same source they are said to be **coherent** and we have : $\boxed{I = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\left(\frac{2\pi\delta}{\lambda}\right)}$ There are interferences.
- In all other cases, the sources are **incoherent** and $I = I_1 + I_2$.

7.2 Applications

Path difference for two coherent sources (Young's slits)

Consider two coherent sources S_1 and S_2 separated by a distance a. We watch the interferences on a screen of vertical axis x at a distance D.

Then $\boxed{\delta = \frac{ax}{D}}$ The **fringe spacing** (distance between two dark fringes) is here $\boxed{i = \frac{\lambda D}{a}}$

With a lens

Add a convergent lens so that F' is on the screen. Now $\boxed{\delta = \frac{ax}{f'}}$ and $i = \frac{\lambda f'}{a}$

Moving the primary source

Consider a source S at a distance D' from two slits that become secondary sources S_1 and S_2 by diffraction. Here S is at a distance x' from the horizontal.

We have $\delta = \frac{ax}{D} + \frac{ax'}{D'}$

Rectangular source

We define the **constrast** $C = \frac{I_{max} - I_{min}}{I_{max} + I_{min}}$. For a rectangular source of size h, $C = \left|\text{sinc}\frac{\pi a h}{\lambda D'}\right|$.

Influence of the spectral width

For a sodium lamp, there is emission in two close wavelengths in the yellow λ_1, λ_2. We study the interferences through YOUNG slits. We define $\sigma_i = \frac{1}{\lambda_i}$. There is interference and $C = |\cos \Delta\sigma\pi\delta|$

7.3 Diffraction gratings

Grating formulas

A **grating** is a repetition of slits each separated from the same distance a.

- In transmission, $\boxed{\delta = a(\sin\theta_{out} - \sin\theta_{in})}$
- In reflexion, $\boxed{\delta = a(\sin r + \sin i)}$

8 Magnetostatic

Definition

The **volumic current vector** $\overrightarrow{j}$ is so that $I = \iint \overrightarrow{j}.\mathrm{d}\overrightarrow{S}$

Biot and Savart's formula (for use in case of absolute despair)

For a circuit $\mathcal{C}$, the magnetic field at a point M is $\overrightarrow{B}(M) = \oint\limits_{P\in\mathcal{C}} \frac{\mu_0 i_P \overrightarrow{\mathrm{d}l_P} \wedge \overrightarrow{PM}}{4\pi PM^3}$

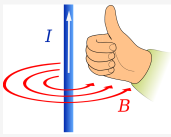
It is the analogue of $\overrightarrow{E}(M) = \iiint\limits_{P\in\mathbb{R}^3} \frac{\rho \mathrm{d}\tau \overrightarrow{PM}}{4\pi\varepsilon_0 PM^3}$.

Symmetries for $\overrightarrow{B}$

$\overrightarrow{B}$ is a **pseudo-vector** :

$\overrightarrow{B}$ is orthogonal to symmetry planes is is parallel to antisymmetry planes.

For the field emitted by a current loop, use the corkscrew law to know the direction of $\overrightarrow{B}$, and remember LORENTZ's force $F = q(\overrightarrow{E} + \overrightarrow{v} \wedge \overrightarrow{B})$



Vector Potential

There exists a **vector potential** $\overrightarrow{A}$ so that $\overrightarrow{B} = \overrightarrow{\mathrm{rot}}(\overrightarrow{A})$

Maxwell-Thomson's equation

- **Integral form** : $\boxed{\oiint\limits_{M\in S} \overrightarrow{B}(M).\mathrm{d}\overrightarrow{S} = 0}$ for all **closed** surfaces S.
- **Local form** : $\boxed{\mathrm{div}\overrightarrow{B} = 0}$ at every point in space.

Ex 32

We consider a current loop of intensity I and of radius R. Prove that along the axis, $\boxed{\overrightarrow{B} = \frac{\mu_0 I R^2}{2(R^2+z^2)^{\frac{3}{2}}} \overrightarrow{u_z}}$ then compute the radius of a field tube in the vicinity of the axis.

First "passage relation"

At the interface of a surface containing a surface current : $\boxed{\overrightarrow{B_-}.\overrightarrow{n} = \overrightarrow{B_+}.\overrightarrow{n}}$

Maxwell-Ampère's equation

- **Integral form** : $\boxed{\oint_{P\in C} \overrightarrow{B}(P).\overrightarrow{\mathrm{d}l_P} = \mu_0 I_{enlaced}}$ for all **closed** circulations C.

  With $I_{enlaced} = \iint_{M\in S} \overrightarrow{j_M}.\overrightarrow{\mathrm{d}S_M}$

- **Local form** : $\boxed{\overrightarrow{\mathrm{rot}}\overrightarrow{B} = \mu_0 \overrightarrow{j}}$ at every point (**in static**).

Example : for a charged wire of intensity I, $\overrightarrow{B} = \dfrac{\mu_0 I}{2\pi r}$

Consequence : $\mathrm{div}\,\overrightarrow{j} = 0$, So we have the **node law**.

Second "passage relation"

At the interface of a surface with a surface current $\overrightarrow{j_s}$, $\boxed{\overrightarrow{B_2} - \overrightarrow{B_1} = \mu_0 \overrightarrow{j_s} \wedge \overrightarrow{n}_{1\to 2}}$

Examples

- "Current tablecloth"
- Infinite Solenoid : $\boxed{\overrightarrow{B} = \mu_0 n I \overrightarrow{u_z}}$ (solenoid of axis $\overrightarrow{u_z}$, loop density n and intensity I).

Magnetic Moment

For a plane circuit of intensity I and of oriented surface $S\overrightarrow{n}$, we define its **magnetic moment** $\boxed{\overrightarrow{M} = IS\overrightarrow{n}}$.

We now consider a magnetic moment $\overrightarrow{M}$ created by a dipole at a distance $r \gg a$ (the size of the dipole).

Reminder : the electrostatic dipole

For an electric dipole moment $\overrightarrow{p}$, the electric potential is (in spherical coordinates) : $\boxed{V = \dfrac{p\cos\theta}{4\pi\varepsilon_0 r^2}}$.

Then since $\overrightarrow{E} = -\overrightarrow{\mathrm{grad}}V$, $E_r = -\dfrac{\partial V}{\partial r} = \dfrac{2p\cos\theta}{4\pi\varepsilon_0 r^3}$ and $E_\theta = -\dfrac{1}{r}\dfrac{\partial V}{\partial\theta} = \dfrac{p\sin\theta}{4\pi\varepsilon_0 r^3}$

For a uniform external field $\overrightarrow{E_{ext}}$, the dipole receives a couple $\boxed{\overrightarrow{\Gamma} = \overrightarrow{p} \wedge \overrightarrow{E_{ext}}}$

The electrostatic potential energy is $\boxed{E_p = -\overrightarrow{p}.\overrightarrow{E_{ext}}}$ ($\overrightarrow{E_{ext}}$ can be nonconstant here)

By analogy :

Magnetic Dipole Formulas

In spherical coordinates, $\boxed{\overrightarrow{B} = \dfrac{\mu_0 M}{4\pi r^3}(2\cos\theta\overrightarrow{u_r} + \sin\theta\overrightarrow{u_\theta})}$

For a uniform external fied $\overrightarrow{B_{ext}}$, $\boxed{\overrightarrow{\Gamma} = \overrightarrow{M} \wedge \overrightarrow{B_{ext}}}$

In all cases, the potential energy is $\boxed{\overrightarrow{E_p} = -\overrightarrow{M}.\overrightarrow{B_{ext}}}$

8.1 TD

Ex 33

We consider a source S moving along the axis x' such that $S_{x'} = vt$. Its light rays go through two YOUNG slits separated from a distance a and disposed symmetrically on each side of the optical axis. The slits are at a distance L from the x' axis. After the slits, a convergent lens of focal f' is disposed so that F' is on the screen.

What do we see on the screen ?

Ex 34

1) We consider an infinite cylinder of radius b_1 and of axis z containing a volumic current $\overrightarrow{j} = j\overrightarrow{u_z}$. Compute the value of $\overrightarrow{B}$ everywhere, and give an instrinsic form.

2) We consider the same cylinder but with a cylindrical cavity of the same axis but of radius $b_2 < b_1$. Give the value $\overrightarrow{B}$ everywhere.

3) We consider the first cylinder but with a cylindrical cavity of radius b_2 and axis z' parallel to z with $zz' = 2a$. Find $\overrightarrow{B}$ inside the cavity in an instrinsic form.

Ex 35

A cylinder of length l and radius a is turning along its axis at a constant angular speed ω. It contains a uniform volumic charge ρ.

Using approximations, compute the created field $\overrightarrow{B}$ everywhere.

Ex 36

By analogy with the electric field, we consider the gravitational field $\overrightarrow{g}$ that satifies GAUSS's theorem :

- $\oiint \overrightarrow{g}.\mathrm{d}\overrightarrow{S} = -4\pi G M_{int}$ (M_{int} being the mass inside the chosen volume)
- $\mathrm{div}\overrightarrow{g} = -4\pi G\mu$ (μ being the volumic density of mass).

We consider the ground to have a uniform volumic density of mass μ. We want to find a hidden ball of gold (volumic mass μ_G) at a depth h and of radius a. What is the variation of the gravitational field at ground level above the gold ?

9 Maxwell's equations

9.1 Generalities

We are no longer in static !

Maxwell's equations - local form

MAXEWELL-GAUSS	$\mathrm{div}\overrightarrow{E} = \dfrac{\rho}{\varepsilon_0}$
MAXEWELL-THOMSON	$\mathrm{div}\overrightarrow{B} = 0$
MAXEWELL-FARADAY	$\overrightarrow{\mathrm{rot}}\overrightarrow{E} = -\dfrac{\partial\overrightarrow{B}}{\partial t}$
MAXEWELL-AMPÈRE	$\overrightarrow{\mathrm{rot}}\overrightarrow{B} = \mu_0\overrightarrow{j} + \varepsilon_0\mu_0\dfrac{\partial\overrightarrow{E}}{\partial t}$

Charge conservation

The charge conservation equation is $\boxed{\dfrac{\partial \rho}{\partial t} + \mathrm{div}\,\vec{j} = 0}$

Integral forms

MAXEWELL-GAUSS	$\displaystyle\oiint \vec{E}.\mathrm{d}\vec{S} = \frac{Q_{int}}{\varepsilon_0}$
MAXEWELL-THOMSON	$\displaystyle\oiint \vec{B}.\mathrm{d}\vec{S} = 0$
MAXEWELL-FARADAY	$\displaystyle\oint \vec{E}.\mathrm{d}\vec{l} = -\frac{\mathrm{d}\phi_B}{\mathrm{d}t}$
MAXEWELL-AMPÈRE	$\displaystyle\oint \vec{B}.\mathrm{d}\vec{l} = \mu_0 I_{enlaced} + \varepsilon_0\mu_0\frac{\mathrm{d}\phi_E}{\mathrm{d}t}$

Potentials

$\boxed{\vec{B} = \overrightarrow{\mathrm{rot}}\,\vec{A}}$ (the **vector potential**) and $\boxed{\vec{E} = -\overrightarrow{\mathrm{grad}}V - \dfrac{\partial \vec{A}}{\partial t}}$

9.2 Field energy

Volumic expressions

If all particles have the same speed, $\boxed{\vec{j} = nq\vec{v}}$ where n is the particle density.

The volumic force applied to the charges by the field is $\boxed{\vec{f_v} = \rho\vec{E} + \vec{j} \wedge \vec{B}}$

The associated volumic power is $\boxed{P_v = \vec{j}.\vec{E}}$

Poynting's theorem

Consider U_{em} the electromagnetic energy and w its volumic counterpart.

The field radiates a power $\boxed{P_{rad} = \displaystyle\oiint \vec{\pi}.\mathrm{d}\vec{S}}$ where $\boxed{\vec{\pi} = \dfrac{\vec{E} \wedge \vec{B}}{\mu_0}}$, POYNTING's vector.

The field gives a power $\boxed{P_J = \displaystyle\iiint \vec{j}\,\vec{E}\,\mathrm{d}\tau}$ to the charged particles by JOULE effect.

POYNTING's theorem in integral form : $\boxed{\dfrac{\mathrm{d}U_{em}}{\mathrm{d}t} = -\displaystyle\oiint \vec{\pi}.\mathrm{d}\vec{S} - \iiint \vec{j}.\vec{E}\,\mathrm{d}\tau}$

In local form : $\boxed{\dfrac{\partial w}{\partial t} = -\mathrm{div}\,\vec{\pi} - \vec{j}.\vec{E} \text{ with } w = \dfrac{\varepsilon_0 E^2}{2} + \dfrac{B^2}{2\mu_0}}$

Examples : electric energy of a condensator : $E_C = \dfrac{Q^2}{2C}$, for an inductance : $E_L = \dfrac{\mu_0 N^2 i^2 S}{2h}$

9.3 EM waves in vacuum

D'Alembert's equation

In vacuum, $\boxed{\overrightarrow{\Delta}\overrightarrow{E} = \dfrac{1}{c^2}\dfrac{\partial^2 \overrightarrow{E}}{\partial t^2}, \quad \overrightarrow{\Delta}\overrightarrow{B} = \dfrac{1}{c^2}\dfrac{\partial^2 \overrightarrow{B}}{\partial t^2} \quad \text{where } \varepsilon_0 \mu_0 c^2 = 1}$

9.4 Approximations

Electric ARQS

ARQS : Approximation for Regimes that are Quasi-Stationary (french acronym).

We consider that in the electric ARQS, $\mu_0 \overrightarrow{j} \ll \varepsilon_0 \mu_0 \dfrac{\partial \overrightarrow{E}}{\partial t}$ and $\overrightarrow{\text{rot}}\overrightarrow{E} \approx \overrightarrow{0}$

Here you can consider the static equations for $\overrightarrow{E}$ (but not $\overrightarrow{B}$). We say that $\overrightarrow{E}$ adapts instantly to the changes in charge. This approximation is very rarely used.

Magnetic ARQS

Here we consider that $\dfrac{\rho}{\varepsilon_0} \approx 0$ and $\mu_0 \overrightarrow{j} \gg \varepsilon_0 \mu_0 \dfrac{\partial \overrightarrow{E}}{\partial t}$.

Here you can consider the static equations for $\overrightarrow{B}$ (but not $\overrightarrow{E}$). We say that $\overrightarrow{B}$ adapts instantly to the changes in current. This approximation is used very commonly.

9.5 Conductors

Drude's model

DRUDE's model is a study of the electrons inside a conductor using classic mechanics, adding a friction force $\overrightarrow{f} = -\alpha \overrightarrow{v}$.

An alternative study is to modelise the friction by impacts between the electrons.

Ohm's law

For a conductor with a current at a frequence $f \ll 10^{14} Hz$, $\boxed{\overrightarrow{j} = \gamma \overrightarrow{E}}$ where γ is the **conductivity** of the conductor.

Electroneutrality in a conductor

In a condutor, $\dfrac{\partial \rho}{\partial t} + \dfrac{\gamma}{\varepsilon_0}\rho = 0$

In the conditions of OHM's law, we can apply the magnetic ARQS inside the condutor.
Examples : resistors in static :

- For a cylindrical resistor, $R = \dfrac{L}{\gamma S}$

- For a hollowed out cylinder $R = \dfrac{\ln \frac{R_2}{R_1}}{2\pi\gamma H}$

Laplace's force

The volumic LAPLACE force is $\overrightarrow{f_v} = \overrightarrow{j} \wedge \overrightarrow{B}$. The small force on a portion $\overrightarrow{\mathrm{d}l}$ is $\boxed{\overrightarrow{\delta f_L} = I\overrightarrow{\mathrm{d}l} \wedge \overrightarrow{B}}$

The resultant on a circuit C is $\boxed{\overrightarrow{F_L} = \oint_C I\overrightarrow{\mathrm{d}l} \wedge \overrightarrow{B_{ext}}}$.

The resulting moment is $\overrightarrow{\Gamma_L} = \oint_{P\in C} \overrightarrow{OP} \wedge \left(I\overrightarrow{\mathrm{d}l}_P \wedge \overrightarrow{B_{ext}}(P)\right)$

The **magnetic moment** of the current loop is $\overrightarrow{M} = IS\overrightarrow{n}$. By analogy with magnetic dipoles :

If $\overrightarrow{B_{ext}}$ varies little at the scale of the circuit, $\boxed{\overrightarrow{F_L} = -\overrightarrow{\mathrm{grad}}E_p = \overrightarrow{\mathrm{grad}}(\overrightarrow{M}.\overrightarrow{B}) \text{ and } \overrightarrow{\Gamma_L} = \overrightarrow{M} \wedge \overrightarrow{B}}$

If $\overrightarrow{B_{ext}}$ is constant, then $\overrightarrow{F_L} = \overrightarrow{0}$.

9.6 TD

Ex 37

We consider two current loops (1) and (2) of radii $R_1 \gg R_2$ with currents I_1 and I_2, disposed along the same axis (Oz) separated by a distance D. Our objective is to find the LAPLACE force applied to (2) by (1) : $\overrightarrow{F_{1\to 2}}$

1) Usin O_1 as origin, give the value of $\overrightarrow{B_1}$ along the axis of the loops.

2) Considering a cylinder of radius r and length $\mathrm{d}z$, find $\overrightarrow{B_1}(r, z)$ close to the axis.

3) Compute $\overrightarrow{F_{1\to 2}}$.

Ex 38

We modelise lightning by a cylinder of radius $a = 2cm$ with a current $I = 50kA$ ($\overrightarrow{j}$ is uniform) with everything in permanent regime. Find the pressure $P(0)$ at the center of the lightning.

Ex 39

We consider an infinite cylinder of radius a, intensity I ($\overrightarrow{j}$ is uniform) and conductivity γ.

1) Find $\overrightarrow{E}$ and $\overrightarrow{B}$.

We now consider a portion of height h of the previous cylinder.

2) Find ϕ the flux of POYNTING's vector on the portion.

3) Knowing that the power associated JOULE's effect is $P_J = RI^2$, find the resistance R using an energetic equation on the cylinder.

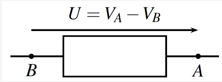
10 Electricity

10.1 General Laws

Tension

The **tension** between points A and B is $u_{AB} = V_A - V_B$

Conventions : for a generator the tension goes in the direction of the current, for a receptor it goes in the opposite direction of the current.



Loop law : the alebric sum of the currents along a loop is 0.

Intensity

The intensity I at a point in the circuit is $I = \dfrac{\mathrm{d}q}{\mathrm{d}t}$.

Node law : the algebric sum of the currents going to a node is 0.

Condensator

A condensator has a capacity C (in Farad) such that $q = Cu$.

For a condensator, $i = C\dfrac{\mathrm{d}u}{\mathrm{d}t}$, and the tension of a condensator is continuous.

Coil

A coil has an inductance L (in Henry).

For a coil, $u = L\dfrac{\mathrm{d}i}{\mathrm{d}t}$

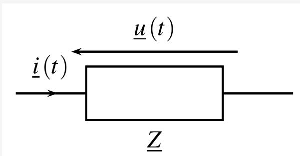
10.2 Complex notation

For a quantity $x(t) = x_m \cos(\omega t + \varphi)$, we associate its complex representation $\underline{x} = \underline{x_m} e^{j\omega t}$ with $\underline{x_m} = x_m e^{j\varphi}$. We have $x = \mathrm{Re}\underline{x}$.

The derivative of $\underline{x}$ according to time is therefore $j\omega\underline{x}$. An antiderivative of $\underline{x}$ according to time is $\dfrac{\underline{x}}{j\omega}$.

Definition

The **complex impedance** $\underline{Z}(\omega)$ of a dipole is $\underline{Z}(\omega) = \dfrac{\underline{u}}{\underline{i}}$.

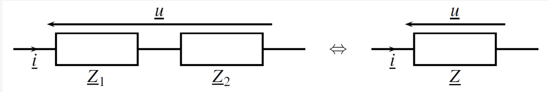


For a resistor, $\underline{Z}_R = R$, for a condensator $\underline{Z}_C = \dfrac{1}{j\omega C}$ and for a coil $\underline{Z}_L = j\omega L$.

10.3 Impedance association

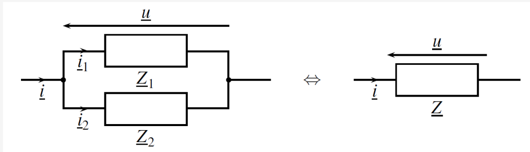
Association in series

For impedances $\underline{Z}_k$ in series, $\underline{Z}_{eq} = \sum_{k=1}^{n} \underline{Z}_k$



Association in parallel

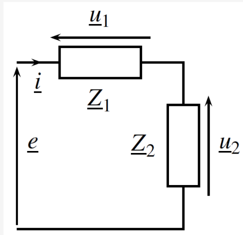
For impedances $\underline{Z}_k$ in parallel, $\dfrac{1}{\underline{Z}_{eq}} = \sum_{k=1}^{n} \dfrac{1}{\underline{Z}_k}$



Tension dividor

The tension $\underline{u}_k$ on $\underline{Z}_k$ is $\underline{u}_k = \underline{e}\dfrac{\underline{Z}_k}{\sum_{a=1}^{n} \underline{Z}_a}$. (The $\underline{Z}_a$ are in series)

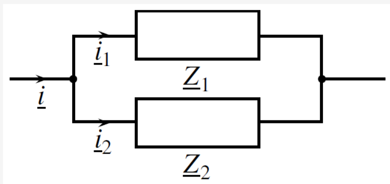
Where e is the tension on all impedances combined.



Current dividor

The current $\underline{i}_k$ on $\underline{Z}_k$ is $\underline{i}_k = \underline{i}\dfrac{\frac{1}{\underline{Z}_k}}{\sum_{a=1}^{n} \underline{Z}_a}$ (The $\underline{Z}_a$ are in parallel)

Where i is the current on all impendances combined.



Milmann's theorem

Consider a node N of potential V_N attached to n impedances $\underline{Z}_k$ of potentials V_k.

Then $V_n = \dfrac{\sum_{k=1}^{n} \frac{V_k}{\underline{Z}_k}}{\sum_{k=1}^{n} \frac{1}{\underline{Z}_k}}$.

Example : Series R,C circuit.

10.4 Resonance

Two examples :

- Resonance in i in a R,L,C circuit
- Resonance in x for a mechanical forced oscillator.

10.5 Filters

Definition

A **filter** is a device that selects certain frequencies. In electricty we consider an entry tension $\underline{e}$ and an exit tension $\underline{s}$.

We define the **transfer function** of a filter by $\underline{H} = \frac{\underline{s}}{\underline{e}} = \frac{s_m}{e_m}$.

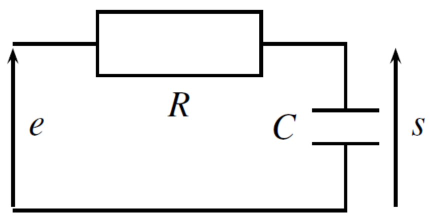
We define the **yield** of the filter as $G = |\underline{H}|$ and the **decibel yield** $G_{dB} = 20 \log G$.

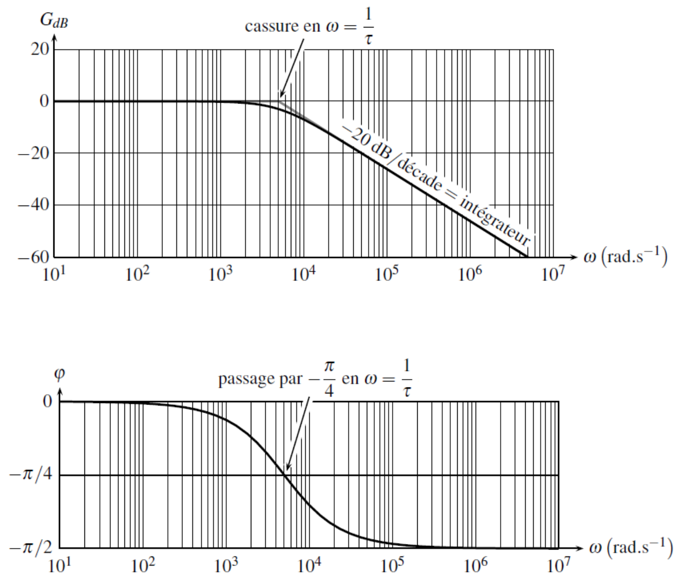
We define the **cutoff pulsation** ω_c so that $G(\omega_c) = \frac{G_{max}}{\sqrt{2}}$.

The **bandwidth** is the interval in ω such that $\frac{G_{max}}{\sqrt{2}} \leq G(\omega) \leq G_{max}$.

Pass-low of the first order

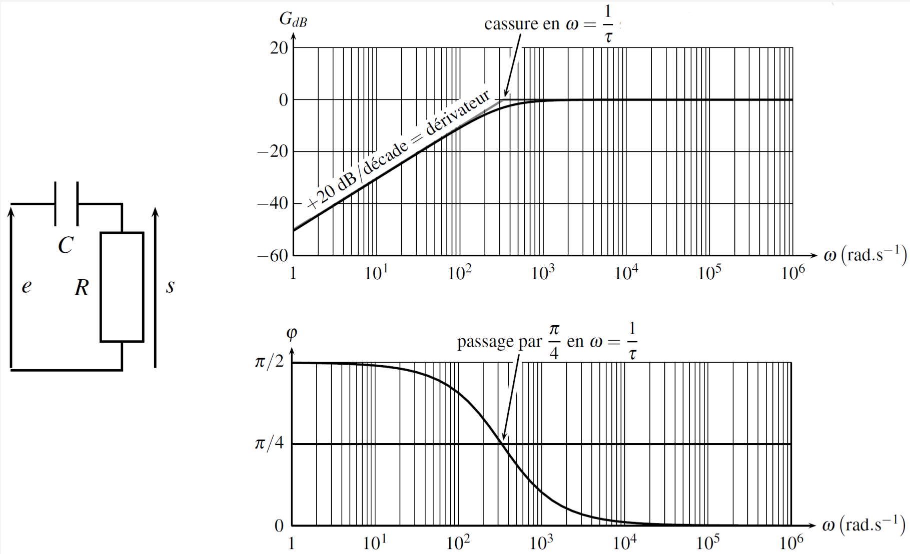
The canonic transfer function is $\boxed{\underline{H} = \frac{A}{1 + jx}}$. For a R,C circuit, $\tau = RC$, $\omega_0 = \frac{1}{\tau}$, $x = \frac{\omega}{\omega_0}$.





Pass-high of the first order

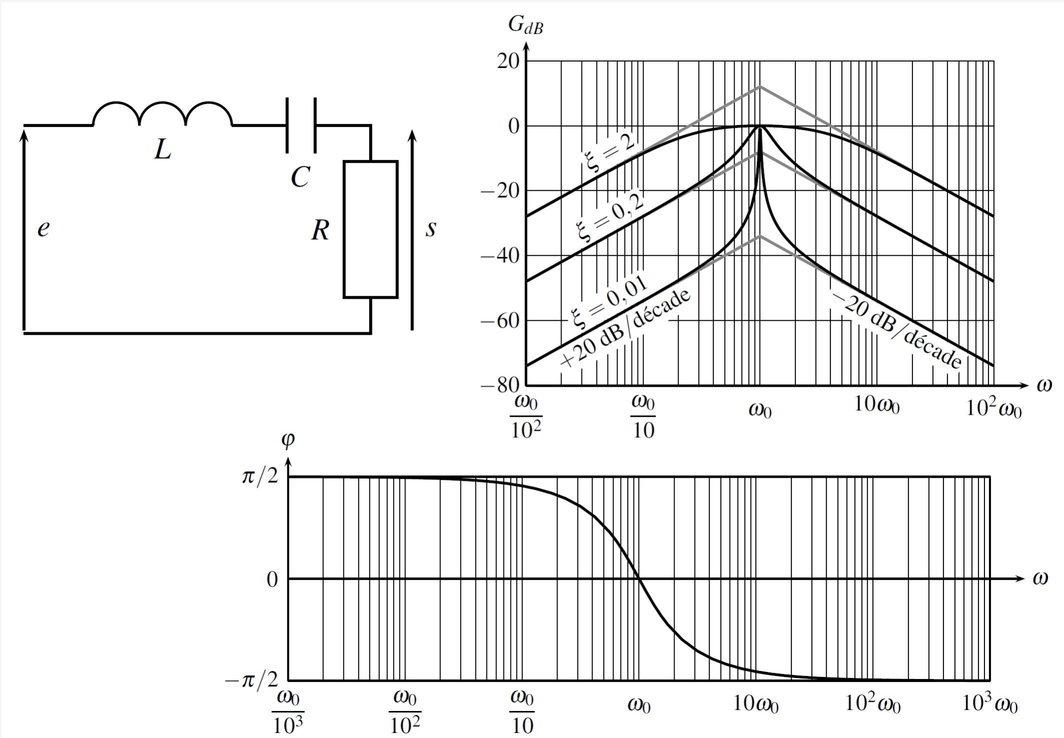
The canonic transfer function is $\boxed{\underline{H} = A\dfrac{jx}{1+jx}}$.



Band-pass

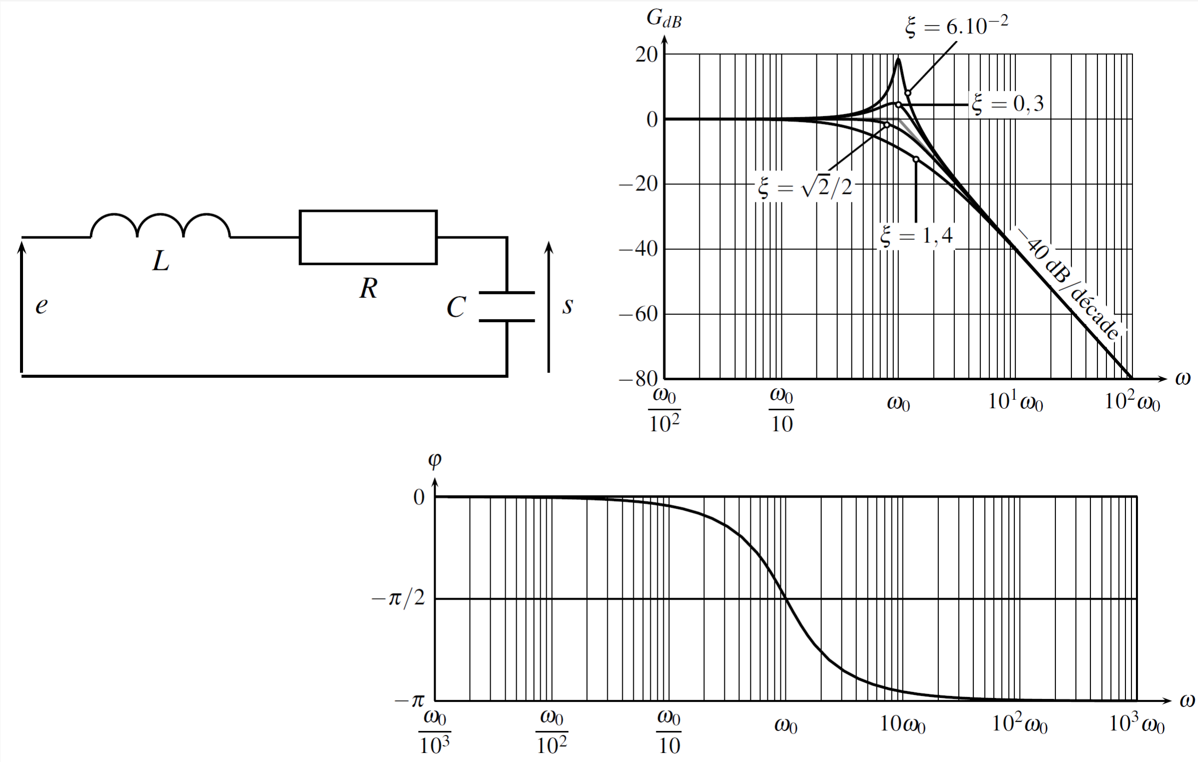
The canonic transfer function is $\boxed{\underline{H} = \dfrac{A}{1+jQ\left(x-\dfrac{1}{x}\right)}}$

For an R,L,C circuit, $\omega_0 = \dfrac{1}{\sqrt{LC}}$, $Q = \dfrac{1}{R}\sqrt{\dfrac{L}{C}} = \dfrac{1}{RC\omega_0} = \omega_0\dfrac{L}{R}$. We define $\xi = \dfrac{1}{2Q}$.



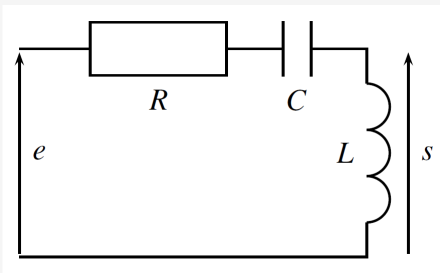
Pass-low of the second order

The canonic transfer function is $\boxed{\underline{H} = A\dfrac{1}{1-x^2+\dfrac{jx}{Q}}}$. Resonance when $Q > \dfrac{\sqrt{2}}{2}$ or $\xi < \dfrac{\sqrt{2}}{2}$



Pass-high of the second order

The canonic transfer function is $\boxed{\underline{H} = A\dfrac{-x^2}{1-x^2+j\dfrac{x}{Q}}}$ Resonance when $Q > \dfrac{\sqrt{2}}{2}$ or $\xi < \dfrac{\sqrt{2}}{2}$



10.6 TD

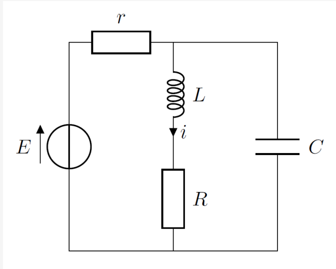
Ex 40

We consider the series circuit E, r, L, R to which we add a condensator C at $t = 0$.

1) Give $i(0^-)$ and $i(0^+)$.

2) Find $\left.\dfrac{\mathrm{d}i}{\mathrm{d}t}\right)_{t=0^+}$.

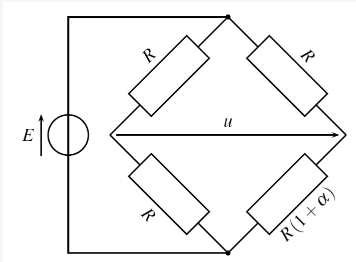
3) Provide a differential equation on i at $t > 0$.



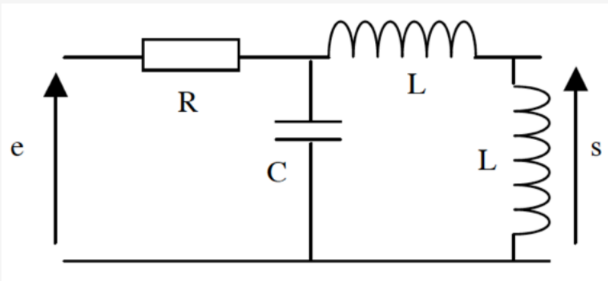
Ex 41

This is a motion sensor. Under mechanical constraint, α varies and we measure a difference in the tension u.

Find an equation linking E, α and u. (*Hint* : consider the potentials at every corner of the square and impose a mass)



Ex 42

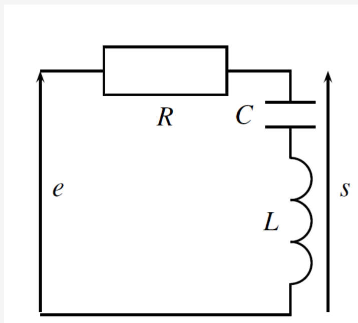


After having given the behaviour at high frequency and low frequency, find the transfer function and put it under the form :

$$\underline{H} = H_0 \frac{j\dfrac{x}{Q}}{1 - x^2 + j\dfrac{x}{Q}} \text{ with } x = \frac{\omega}{\omega_0}.$$ Give the values of ω_0, Q and H_0.

Ex 43

Study the following filter :



11 Induction

11.1 Effect of Laplace's force on circuits

Two examples : Laplace's rails and a rotating current loop.

11.2 Neumann Induction

Neumann induction is when the circuit is fix but the magnetic field varies.

Consider a current loop determining a surface S and $\phi = \iint_S \overrightarrow{B}.\mathrm{d}\overrightarrow{S}$.

If ϕ varies, then there is an induced current inside the loop. This current causes a Laplace force and an induced field $\overrightarrow{B_i}$

Lenz's law

Induction opposes its origin : the induced currents cause a Laplace force that opposes the cause of the induction.

Faraday's law

If the flux ϕ of the circuit varies, then an electromotive force is inducted in the circuit $\boxed{e = -\frac{\mathrm{d}\phi}{\mathrm{d}t}}$

Auto-induction

Consider a circuit with an intensity i. There is a self-field $\overrightarrow{B_s}$ created because of the current.

This self-field has a flux on the circuit, ϕ_s with $\boxed{\phi_s = Li}$.

L is a constant (property of the circuit) called the **inductance** of the circuit.

Therefore all circuits have an auto-induced electromotive force $e_s = -L\frac{\mathrm{d}i}{\mathrm{d}t}$

Mutual induction

Consider two coils (1) and (2). They emit self-fields $\overrightarrow{B}_1$ and $\overrightarrow{B}_2$ that have fluxes $\phi_{1\to 2}$ and $\phi_{2\to 1}$ on each other.

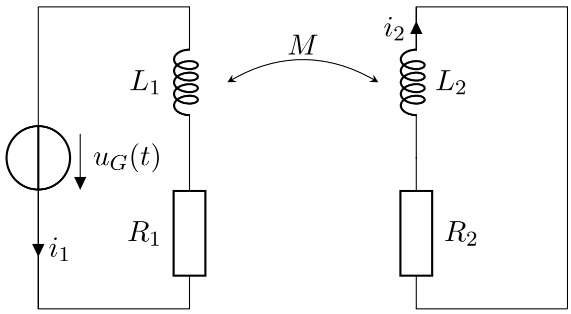
There are therefore induced electromotive forces. In (2) : $e_{1\to 2} = -\frac{\mathrm{d}\phi_{1\to 2}}{\mathrm{d}t} = -M_{1\to 2}\frac{\mathrm{d}i_1}{\mathrm{d}t}$

And in (1) : $e_{2\to 1} = -\frac{\mathrm{d}\phi_{2\to 1}}{\mathrm{d}t} = -M_{2\to 1}\frac{\mathrm{d}i_2}{\mathrm{d}t}$

Neumann's theorem is $\boxed{M_{1\to 2} = M_{2\to 1}}$

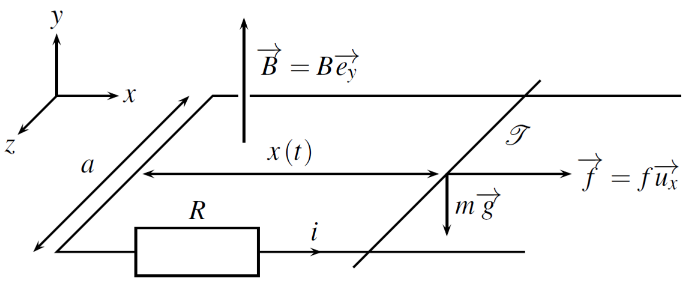
Example : a coil inside another. In that case, $M^2 = L_1 L_2$

Energetic study of mutual inductances :

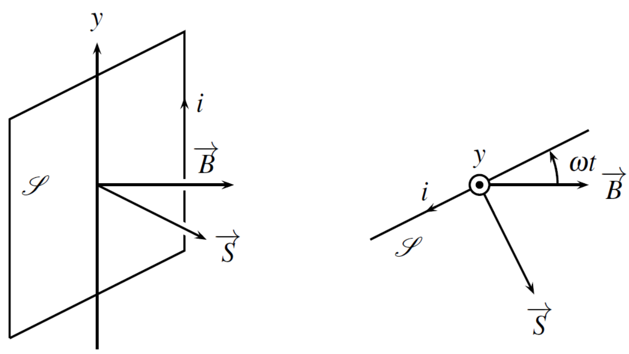


11.3 Lorentz induction

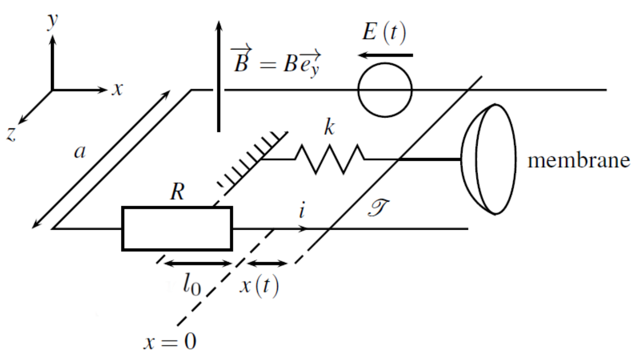
Laplace's rails, an example of the law $\boxed{P_{\text{Lapalce}} + P_{\text{elec}} = 0}$



Other example : rotating current loop, illustrating $\boxed{\vec{\Gamma}_{ext} + \vec{\Gamma}_{\text{Laplace}} = \vec{0}}$:



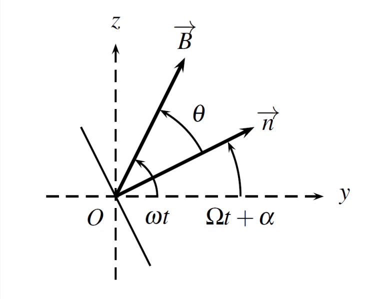
Speakers :



11.4 TD

Ex 44

Synchrone Motor
A motor is made of a "stator" : creating a field $\overrightarrow{B} = B_o\overrightarrow{u(t)}$ and a "rotor" : a current loop with N loops, of surface S and intensity I, rotating at a constant speed Ω with an initial angle α and a normal vector $\overrightarrow{n}$.



Find $\overrightarrow{\Gamma_m}(t)$ the moment of the electromagnetic forces on the rotor. Find its average according to time $\left\langle\overrightarrow{\Gamma_m}\right\rangle$. Why is this called a synchrone motor ?

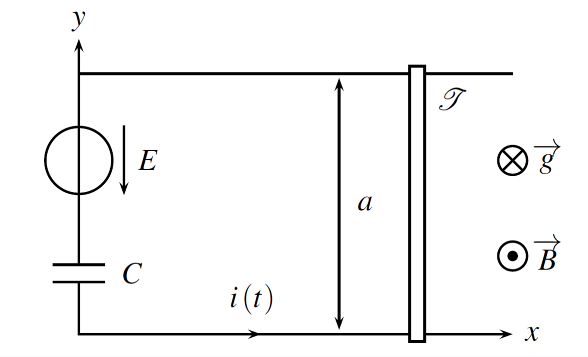
Ex 45

Asynchrone motor
We consider the same motor as in Ex 1, but the stator is rotating at ω' and the rotor at $\omega < \omega'$. Furthermore, the rotor is now a circuit of resistance R and self-inductance L.

Find $\left\langle\overrightarrow{\Gamma}\right\rangle$ the average moment of the electromagnetic forces on the rotor. Why is this called an asynchrone motor ? What value of $\omega' - \omega$ gives the best efficiency for the motor ?

Ex 46

Find the speed $\overrightarrow{v}(t)$ of the rail :



12 Electromagnetic Waves

12.1 EM Waves in vacuum

Definition

A **plane** wave of axis along $\overrightarrow{u}$ is a wave for which $\overrightarrow{E}$ and $\overrightarrow{B}$ are uniform in any plane perpendicular to $\overrightarrow{u}$.

In this case, the fields are under the form $\overrightarrow{E}(\overrightarrow{r}.\overrightarrow{u},t)$, $\overrightarrow{B}(\overrightarrow{r}.\overrightarrow{u},t)$

A **plane progressive wave (PPW)** going in the direction $\overrightarrow{u}$ is of the form $\overrightarrow{f}(\overrightarrow{r}.\overrightarrow{u} - ct)$

Structure of the PPW in vacuum

In vacuum, $\overrightarrow{u}, \overrightarrow{E}, \overrightarrow{B}$ are directly orthogonal, $c\overrightarrow{B} = \overrightarrow{u} \wedge \overrightarrow{E}$, $\quad \overrightarrow{E} = c\overrightarrow{B} \wedge \overrightarrow{u}$ and $\|\overrightarrow{E}\| = c\|\overrightarrow{B}\|$

PPHW

A wave is **plane progressive harmonic** when $\overrightarrow{E}$ can be written under the form :

$\overrightarrow{E} = \begin{pmatrix} E_x^0\cos(\omega t - \overrightarrow{k}.\overrightarrow{r} + \varphi_x) \\ E_y^0\cos(\omega t - \overrightarrow{k}.\overrightarrow{r} + \varphi_y) \\ E_z^0\cos(\omega t - \overrightarrow{k}.\overrightarrow{r} + \varphi_z) \end{pmatrix}$. With the complex notation : $\boxed{\underline{\overrightarrow{E}} = \underline{\overrightarrow{E_0}} e^{i(\omega t - \overrightarrow{k}.\overrightarrow{r})}}$ ($\overrightarrow{k} = k\overrightarrow{u}$)

Computation rules : $\dfrac{\partial}{\partial t} \leftrightarrow \times i\omega, \quad \overrightarrow{\nabla} = -i\overrightarrow{k}$

Dispersion relation : $\boxed{\omega^2 = k^2c^2}$

Energetics

For a PPHW of direction $\overrightarrow{u_z}$ in vacuum, $\langle \overrightarrow{\pi} \rangle = \dfrac{c\varepsilon_0 E_0^2}{2}$, and $\dfrac{\partial w}{\partial t} + \text{div}\,\overrightarrow{\pi} = 0$

Method : for computing the average of fg when f and g are of the same pulsation, $\boxed{\langle fg \rangle = \dfrac{1}{2}\text{Re}\left(\underline{f}\underline{g}^*\right)}$ ($\underline{g}^*$ is the conjugate of $\underline{g}$)

Polarisation

Consider a PPHW of direction $\overrightarrow{u_z}$. Then $\overrightarrow{E} = E_x^0\cos(\omega t - kz + \varphi_x)\overrightarrow{u_x} + E_y^0\cos(\omega t - kz + \varphi_y)\overrightarrow{u_y}$.

- There is **linear polarisation** when $\overrightarrow{E}$ describes a segment :
  when $E_x^0 = 0$ or $E_y^0 = 0$ or $\varphi_x = \varphi_y[\pi]$
- There is **circular polarisation** when $\overrightarrow{E}$ describes a circle :
  when $E_x^0 = E_y^0$ and $\varphi_x = \varphi_y \pm \dfrac{\pi}{2}[2\pi]$
- In other cases, there is **elliptical polarisation** and $\overrightarrow{E}$ describes an ellipsis.

12.2 EM waves in a diluted plasma

Definition

The **dielectric constant** ε_r of a medium is value (often complex and depending on ω, the pulsation of the waves), is so that the **permittivity** of the medium is $\boxed{\varepsilon = \varepsilon_r\varepsilon_0}$.

Definition

A **plasma** is an ionised medium (with free electrons and ions). It is said to be diluted when the density of particles n is small enough to neglect the interactions between charges.

Properties of the EM waves in the diluted plasma

For a transverse PPHW in a diluted plasma :

- There is a complex volumic current vector $\underline{\overrightarrow{j}} = -ne\underline{\overrightarrow{v}}$
- There is a complex conductivity $\underline{\gamma} = \dfrac{ne^2}{i\omega m_e}$
- Ohm's law is satisfied : $\boxed{\underline{\overrightarrow{j}} = \underline{\gamma}\underline{\overrightarrow{E}}}$
- $\boxed{\rho = 0}$ unless if $\omega = \omega_p = \sqrt{\dfrac{ne^2}{m_e\varepsilon_0}}$ the plasma pulsation.
- The propagation relation is $\boxed{\overrightarrow{\Delta}\overrightarrow{E} = \dfrac{1}{c^2}\dfrac{\partial^2\overrightarrow{E}}{\partial t^2} + \dfrac{\omega_p^2}{c^2}\overrightarrow{E}}$
- The dispersion equation is $\boxed{k^2c^2 = \omega^2 - \omega_p^2}$, the medium is dispersive.

The solutions of the propagation relation are of the form $\overrightarrow{A}e^{i(\omega t - \underline{k}z)}$ (propagation along $\overrightarrow{u_z}$).

- If $\omega < \omega_p$: $\underline{\overrightarrow{k}} = \dfrac{i}{\delta}\overrightarrow{u_z}$ and $\underline{\overrightarrow{E}} = \underline{\overrightarrow{E_0}}e^{i\omega t}e^{-\frac{z}{\delta}}$, there is no propagation : only an evanescent wave.
- If $\omega > \omega_p$: $k = \pm\sqrt{\dfrac{\omega^2 - \omega_p^2}{c}}$. PPHM can pass through.

12.3 Wave packets

Plane progessive harmonic waves don't actually exist. What we see in nature are **superpositions** of PPHW called **wave packets**.

Speed of the wave packet

The **phase speed** is the speed of one wave in the packet that has a pulsation ω and wave vector of norm k : $\boxed{v_\varphi = \dfrac{\omega}{k}}$

The **group speed** is the speed of the packet : $\boxed{v_G = \dfrac{\mathrm{d}\omega}{\mathrm{d}k}}$

12.4 EM Waves in conductors

Conductors

We consider the conductor for frequencies $f \ll 10^{14} Hz$:

In particular we have the M-ARQS :$\| \overrightarrow{j} \| \gg \left\| \varepsilon_0 \frac{\partial \overrightarrow{E}}{\partial t} \right\|$ and OHM's law.

- There is a complex volumic current vector $\underline{\overrightarrow{j}} = -ne\underline{\overrightarrow{v}}$
- There is a conductivity $\underline{\gamma} = \frac{ne^2}{i\omega m_e + \alpha} \approx \frac{ne^2}{\alpha}$
- OHM's law is satisfied : $\boxed{\underline{\overrightarrow{j}} = \gamma \underline{\overrightarrow{E}}}$
- $\boxed{\rho = 0}$ at the forced sinusoidal regime (RSF in French).
- The propagation relation is $\boxed{\overrightarrow{\Delta}\overrightarrow{E} = \mu_0 \gamma \frac{\partial \overrightarrow{E}}{\partial t}}$
- The dispersion equation is $\boxed{\underline{k}^2 = -i\omega\gamma\mu_0}$.
- EM waves polarised on $\overrightarrow{u_x}$ going in the direction $\overrightarrow{u_z}$ are $\underline{\overrightarrow{E}} = \underline{E_0} e^{-z/\delta} e^{i(xt - z/\delta)} \overrightarrow{u_x}$

Relexion on an EM wave on a conductor

Consider an incident wave $\underline{\overrightarrow{E_i}} = \underline{E_0} e^{i(\omega t - kz)} \overrightarrow{u_x}$ going towards $+\overrightarrow{u_z}$ (the normal of the surface).

- There is a **reflected wave** $\underline{\overrightarrow{E_r}} = \underline{r}\,\underline{E_0} e^{i(\omega t + kz)} \overrightarrow{u_x}$ goind towards $-\overrightarrow{u_z}$.

  $\underline{r}$ is the **reflexion coefficient in amplitude**.
- There is a **transmitted wave** $\underline{\overrightarrow{E_t}} = \underline{t}\,\underline{E_0} e^{i(\omega t - \underline{\alpha} z)} \overrightarrow{u_x}$.

  $\underline{t}$ is the **transmission coeffecicient in amplitude**.

  $\underline{\overrightarrow{\alpha}}$ is the wave vector of the wave inside the conductor.

The **coefficients in intensity** are $R = \frac{\langle \overrightarrow{\pi_r} \rangle . (-\overrightarrow{u_z})}{\langle \overrightarrow{\pi_i} \rangle . \overrightarrow{u_z}}$ and $T = \frac{\langle \overrightarrow{\pi_t} \rangle . \overrightarrow{u_z}}{\langle \overrightarrow{\pi_i} \rangle . \overrightarrow{u_z}}$

Ideal conductor

In an **ideal conductor**, $\overrightarrow{E} = \overrightarrow{0}$, $\overrightarrow{B} = \overrightarrow{0}$, $\rho = 0$, $\gamma \longrightarrow +\infty$ and $\overrightarrow{j} = \overrightarrow{0}$.

Here $\underline{r} = -1$ and $\underline{t} = 0$.

12.5 EM waves in a cavity

Confination in the propagation's direction

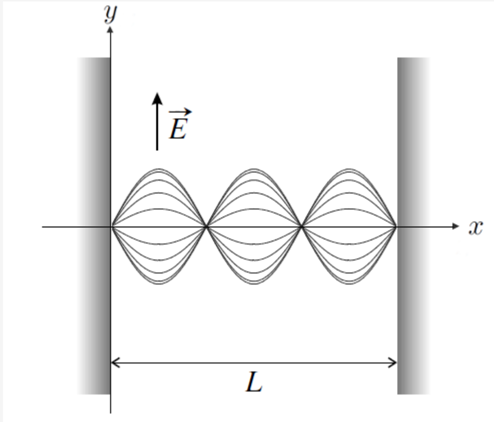
We consider a cavity of length L along $\overrightarrow{u_x}$, the direction of propagation. For $x \leq 0$ and $x \geq L$ we have ideal conductors.

We solve the propagation equation for a field of the form $\overrightarrow{E}(x,t) = f(x)g(t)\overrightarrow{u_y}$.

We obtain **stationary waves** :

$$\overrightarrow{E} = E_0 \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi c}{L}t + \varphi\right)\overrightarrow{u_y}$$

There is resonance in ω as it is quantified : $\omega = \dfrac{n\pi c}{L}$



Wave guide

We consider a field $\underline{E}\overrightarrow{u_y}$ going in the direction $\overrightarrow{u_z}$ inside a cavity between two ideal conductors that are in the semi-planes $x \leq 0$ and $x \geq L$.

We look for solutions of the form $\underline{E} = f(x)e^{i(\omega t - kz)}$

The solutions are $\overrightarrow{\underline{E}} = E_0 \sin\left(\dfrac{n\pi x}{L}\right) e^{i(\omega t - kz)\overrightarrow{u_y}}$

The dispertion equation is $\omega^2 - \omega_c^2 = k^2c^2$ with $\omega_c = \dfrac{n\pi c}{L}$. The wave passes only if $\omega > \omega_c$

We have $v_\varphi = \dfrac{\omega c}{\sqrt{\omega^2 - \omega_c^2}}$ and $v_\varphi v_G = c^2$ (for $\omega > \omega_c$)

This is the same behaviour as in a plasma.

12.6 TD

Ex 47

We consider a field $\underline{\overrightarrow{E}} = E(r)e^{i(\omega t - kr)}\overrightarrow{u_z}$ caused by a source along Oz (cylindrical coordinates), evolving in vacuum.

1) Using the formula $\overrightarrow{\text{rot}}\overrightarrow{A} = \begin{pmatrix} \dfrac{1}{r}\dfrac{\partial A_z}{\partial \theta} - \dfrac{\partial A_\theta}{\partial z} \\ \dfrac{\partial A_r}{\partial z} - \dfrac{\partial A_z}{\partial r} \\ \dfrac{1}{r}\left(\dfrac{\partial (rA_\theta)}{\partial r} - \dfrac{\partial A_r}{\partial \theta}\right) \end{pmatrix}$, compute $\underline{\overrightarrow{B}}$.

2) Determine $\overrightarrow{\pi}$ and then $\langle \overrightarrow{\pi} \rangle$.

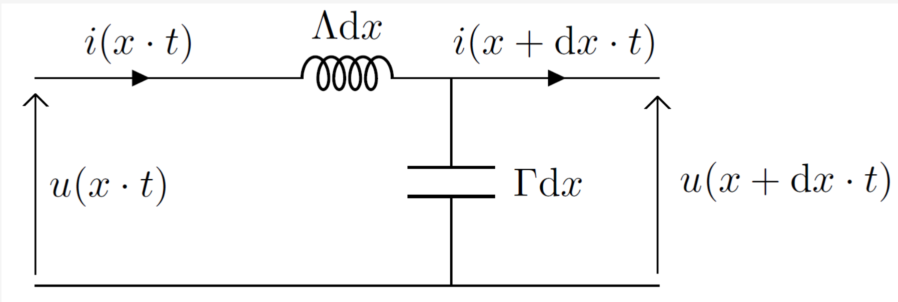
3) Find the power P radiated through a cylinder of radius r and height h. Explain why P doesn't depend on r, and prove that $E(r) = \dfrac{a}{\sqrt{r}}$ with a a constant.

4) Give the values of $\underline{\overrightarrow{E}}$ and $\underline{\overrightarrow{B}}$ when $r \gg \lambda$ and determine the wave structure.

5) Using $\Delta U = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial U}{\partial r}\right)$ if $U = U(r, t)$, find the dispertion equation.

Ex 48

We consier a coaxial cable with a capacity Γ per unit of length and an inductance Λ per unit of length. We modelie the cable locally by the following circuit :



Determine the propagation equation for u and i, as well as the disperstion equation and the characteristic impedance $\underline{Z_C} = \dfrac{\underline{u}}{\underline{i}}$

Ex 49

We consider a string with a mass μ per unit of length. The string is along $\overrightarrow{u_x}$ and can move up and down along $\overrightarrow{u_y}$. We note the tension $\overrightarrow{T}(x)$. We neglect the weight for questions 1) and 2).

1) Prove that $\overrightarrow{T}$ is continuous, we'll consider it to be constant. Find the propagation equation of a wave $u(x, t)$ going along $\overrightarrow{u_x}$ ($u(x, t)$ is the value of y at (x, t)). Give the celerity c of the waves and their expression.

2) We now impose $u(0, t) = U_0 \cos(\omega t)$ and $u(L, t) = 0$. Find the expression of the wave depending on ω.

3) Give the propagation equation and the dispersion equation without neglecting the weight. Is there dispersion ? Can waves still pass ?

Proof of Descartes's laws in geometric optics

We consider an interface at the plane $x = 0$ between two mediums 1) and 2) of indexes n_1 and n_2. We consider both mediums to be abstent of charges and we neglect surfacic charges at the interface.

We consider an incident PPHW $\underline{\overrightarrow{E_i}} = \underline{\overrightarrow{E_i^0}} e^{j(\omega_i t - \overrightarrow{k_i}.\overrightarrow{r})}$ passing by the origin with an angle i with $\overrightarrow{u_x}$ (the normal to the dioptre). We have therefore $k_i = \|\overrightarrow{k_i}\| = \dfrac{\omega_i n_1}{c}$.

There is a reflected PPHW $\underline{\overrightarrow{E_r}} = \underline{\overrightarrow{E_r^0}} e^{j(\omega_r t - \overrightarrow{k_r}.\overrightarrow{r})}$ with $k_r = \|\overrightarrow{k_r}\| = \dfrac{\omega_r n_1}{c}$. It makes an angle r with Ox.

There is a transmitted PPHW $\underline{\overrightarrow{E_t}} = \underline{\overrightarrow{E_t^0}} e^{j(\omega_t t - \overrightarrow{k_t}.\overrightarrow{r})}$ with $k_t = \|\overrightarrow{k_t}\| = \dfrac{\omega_t n_2}{c}$. It makes an angle i' with Ox.

1) Prove that all the waves are at the same pulsation, that the wavevectors $\overrightarrow{k}$ satisfy $\overrightarrow{k_i}.\overrightarrow{u_y} = \overrightarrow{k_r}.\overrightarrow{u_y} = \overrightarrow{k_t}.\overrightarrow{u_y}$ and that they are in the same plane.

2) We now note $k_1 = \dfrac{\omega n_1}{c}$ and $k_2 = \dfrac{\omega n_2}{c}$.

Prove that $|r| = |i|$, that $\overrightarrow{k_i} = -\overrightarrow{k_r}$ and that $n_1 \sin(i) = n_2 \sin(i')$.

3) We consider the case $i > \text{Arcsin}\left(\dfrac{n_2}{n_1}\right)$ and $n_2 < n_1$. Can a light ray pass through the diopter ?

Find the transmitted wave under the form $\underline{\overrightarrow{E_t}} = \underline{\overrightarrow{E_t^0}} e^{j(\omega_t t - \underline{\overrightarrow{k_t}}.\overrightarrow{r})}$ with $\underline{k_t}$ complex. What type of wave is it ?

Give the numerical value of the characteristic propagation distance δ with ordinary parameters.

13 Fluid Mechanics

13.1 Static of Fluids

Pressure Force

The volumic force $\overrightarrow{\varphi}$ attached to a force $\overrightarrow{F}$ is $\overrightarrow{\varphi} = \dfrac{\overrightarrow{F}}{\mathrm{d}\tau}$.

The elementary pressure force is $\boxed{\delta\overrightarrow{F_P} = -P\mathrm{d}\overrightarrow{S}}$

The volumic pressure force is $\boxed{\overrightarrow{\varphi_P} = -\overrightarrow{\mathrm{grad}}P}$

The resultant of the pressure forces on a volume V is therefore $\overrightarrow{F} = \displaystyle\iiint_V -\overrightarrow{\mathrm{grad}}P \; \mathrm{d}\tau$

If the pressure is uniform, then $\overrightarrow{F} = \overrightarrow{0}$: the force on a closed surface is in this case $\overrightarrow{0}$.

Static of fluids

Inside a static fluid, $\boxed{\overrightarrow{\mathrm{grad}}P = \sum_i \overrightarrow{\varphi_i}}$ (the φ_i are the forces that are not pressure forces)

Archmedeses' theorem

The resulting pressure force on a object of volume V and density μ subject only to pressure and its weight is $\boxed{\overrightarrow{\pi_A} = -\mu V \overrightarrow{g}}$

Its point of application is the center of gravity of the solid.

Example : Resulting pressure force on the lateral surface of a submerged cone.

To find the point of application A of a pressure force, use $\displaystyle\iint_{M \in S} \overrightarrow{AM} \wedge P(M)\mathrm{d}\overrightarrow{S} = \overrightarrow{0}$

13.2 Flow rate and conservation laws

Definition

A **current line** is a line tangent to the speed field.

Since $\overrightarrow{v}$ and $\overrightarrow{\mathrm{d}l}_{line}$ are colinear, $\dfrac{\mathrm{d}x}{v_x} = \dfrac{\mathrm{d}y}{v_y} = \dfrac{\mathrm{d}z}{v_z}$.

Given a circulation Γ, a **current tube** is the tube formed by the current lines passing by Γ.

Mass current and mass flow rate

For a fluid of volumic mass $\mu(M,t)$ and of speed field $\overrightarrow{v}(M,t)$:

The **mass current vector** is $\boxed{\overrightarrow{j_m} = \mu\overrightarrow{v}}$

The **mass flow rate** across a section S is $\boxed{D_m = \displaystyle\iint_S \overrightarrow{j_m}.\mathrm{d}\overrightarrow{S}}$ in kg/s

Mass conservation

The local mass conservation equation is $\boxed{\dfrac{\partial \mu}{\partial t} = -\mathrm{div}\overrightarrow{j_m}}$

For a volume of external surface S, the global equation is $\boxed{\dfrac{\mathrm{d}m}{\mathrm{d}t} = -\displaystyle\oiint_S \overrightarrow{j_m}.\mathrm{d}\overrightarrow{S}}$

Volumetric flow rate (or "discharge")

The **volumetric flow rate** accros a surface S is $D_v = \displaystyle\iint_S \overrightarrow{v}.\mathrm{d}\overrightarrow{S}$ in m^3/s

Stationary Flow

There is **stationary flow** when all variables are independent on time.

$\boxed{\text{In this case we have } \mathrm{div}\overrightarrow{j_m} = 0 \text{ which gives the node law for mass flow rates}}$

Homogenous and incompressible flow

There is **homogenous and incompressible flow** when $\mu(M,t) = cte$.

In this case we have div$\overrightarrow{v} = 0$ which gives the node law for volumetric flow rates

13.3 Contact action on a flowing fluid

Viscosity

For a fluid of viscosity η (in "Poiseuille $Pl \equiv kg.m^{-1}.s^{-1}$)

For a flow towards $\overrightarrow{+u_z}$ with $\overrightarrow{v} = v(x,t)\overrightarrow{u_z}$, the volumic visocity force is : $\varphi_v = \eta \dfrac{\partial^2 v}{\partial x^2}(x,t)\overrightarrow{u_z}$

The local force is $\delta\overrightarrow{F} = \eta \mathrm{d}y\mathrm{d}z \left(\dfrac{\partial v}{\partial x}(x+\mathrm{d}x,t) - \dfrac{\partial v}{\partial x}(x,t)\right)\overrightarrow{u_z}$

For a cylindrical flow along $+\overrightarrow{u_z}$ with $\overrightarrow{v} = v(r,t)\overrightarrow{u_z}$:

The local force is $\delta\overrightarrow{F} = \eta(r+\mathrm{d}r)\mathrm{d}\theta\mathrm{d}z\dfrac{\partial v}{\partial r}(r+\mathrm{d}r,t)\overrightarrow{u_z} - \eta r\mathrm{d}\theta\mathrm{d}z\dfrac{\partial v}{\partial r}(r,t)\overrightarrow{u_z}$

The volumic viscosity force is therefore $\overrightarrow{\varphi_v} = \eta\dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial v}{\partial r}\right)$

In general, $\boxed{\overrightarrow{\varphi_v} = \eta\overrightarrow{\Delta}\overrightarrow{v}}$

Method : in practice, either use the general formula (for simple coordinate systems) or prove the local force's expression.

Adherence conditions

Consider an interface between two fluids at the local plane $z = 0$. We consider $\overrightarrow{u_x}$ the normal to the surface. We consider each $\overrightarrow{v_i} = v_i(x)\overrightarrow{u_z}$

The continuity of the speed field imposes $v_1(0) = v_2(0)$ (continuity of $\overrightarrow{v}.\overrightarrow{u_z}$)

The reciprocity of the viscosity forces imposes $\eta_1\dfrac{\mathrm{d}v_1}{\mathrm{d}x}(0) = \eta_2\dfrac{\mathrm{d}v_2}{\mathrm{d}x}(0)$

Examples : Couette and Poiseuille plane flows, Poiseuille cylindrical flow.

13.4 Homogenous incompressible flows in a pipe

In this part we consider $\mu = cte$.

Definition

The **average speed** in a pipe of section S is $\boxed{U = \dfrac{D_v}{S}}$

Example : for a Poiseuille flow on a distance L and a pressure difference $\Delta P = P_1 - P_2$, $U = \dfrac{\Delta P R^2}{8\eta L}$

Momemtum diffusion current vector

We consider a speed field $v(x,t)\overrightarrow{u_z}$. We define the **cinematic viscosity** $\boxed{\nu = \frac{\eta}{\mu}}$

The **momemtum diffusion current vector** is $\boxed{\overrightarrow{j_{p,diff}} = -\nu\overrightarrow{\text{grad}}(\mu v_z)}$

We have the momentum diffusion equation : $\boxed{\nu\frac{\partial^2(\mu v_z)}{\partial x^2} = \frac{\partial(\mu v_z)}{\partial t}}$

Momentum convection current vector

For a speed field $v(x,t)\overrightarrow{u_z}$, the **momemtum convection current vector** is $\overrightarrow{j_{p,conv}} = \mu v_z\overrightarrow{v}$

Reynold's number

Reynold's number is a quantity that determines the type of flow.

Intuitively, $Re = C\frac{\|\overrightarrow{j_{p,conv}}\|}{\|\overrightarrow{j_{p,diff}}\|}$. We define $\boxed{Re = \frac{Ud}{\nu}}$ where U is the average speed, d the diameter of the pipe and ν the cinematic viscosity.

When $Re \ll 2000$, diffusion dominates and the flow is **laminar**.

When $Re \gg 2000$, convection dominates and the flow is **turbulent**.

Flows with small Re

The Hagen-Poiseuille law is $\boxed{D_v R_h = P_1 - P_2}$ For a change of pressure from P_1 to P_2.

The **hydraulic resistance** of a cylinder of radius R, length L with a fluid of viscosity η is $\boxed{R_h = \frac{8\eta L}{\pi R^4}}$

13.5 Macroscopic Equations

Definition

An **ideal flow** is a flow without diffusive behaviours : no viscous forces, no thermal diffusion and thermodynamic irreversibility.

Bernouilli's equation

For an incompressible, homogenous, stationary and ideal flow : $\boxed{P + \mu gz + \frac{1}{2}\mu v^2 = K_{CL}}$ where K_{CL} is a constant depending on the current line.

Definition

The **total pressure** is $P_{tot} = P + \mu gz + \frac{1}{2}\mu v^2$.

The **static pressure** is $P_s = P + \mu gz$

The **dynamic pressure** is $P_d = \frac{1}{2}\mu v^2$

Examples :

- VENTURI effect : if the speed locally increases then the pressure locally decreases.
- COANDA effect : if you blow air on a ping-pong ball with a hair drier, it will draw the ball in rather than push it away.
- VENTURI's rate meter.
- PITOT's tube : a tool for measuring the speed of a fluid.
- TORICELLI's formula : for a leaking container of height h, $v \approx \sqrt{2gh}$

Head loss ("perte de charge")

We define the **head** as $\dfrac{P_{tot}}{\mu g}$.

If the flow isn't ideal, the can be a head loss between two points on the same current line. There are two types of head losses :

- **Regular head loss** : loss of energy due to the pipe's rugosity.

  In practice, we use a pipe's **regular head loss coefficient** $\boxed{\lambda = \dfrac{2d\Delta P_{tot}}{\mu L U^2}}$

- **Singular head loss** : loss of energy due ot a local change in the pipe's geometry.

  We use the singularity's **singular head loss coefficient** $\boxed{\zeta = \dfrac{2\Delta P_{tot}}{\mu U^2}}$.

  For an abrupt increase in section from s to S, $\zeta = \left(1 - \dfrac{s}{S}\right)^2$, for a pipe bend $\zeta \in [0.45, 1.3]$

Reminder : the power given by an operator is $\boxed{P = D_v[P + \mu g z + \frac{1}{2}\mu v]_e^s}$

Mechanical theorems

The momentum of the fluid within a system Σ is $\overrightarrow{p} = \iiint\limits_{\Sigma} \overrightarrow{v}(M,t)\mu(M,t)\mathrm{d}\tau$

For a *closed system* Σ^*, the **Kinetic Resultant Theorem** reads $\boxed{\dfrac{\mathrm{d}\overrightarrow{p}^*}{\mathrm{d}t} = \sum_i \overrightarrow{F_i}}$

The angular momentum of the fluid within Σ is $\overrightarrow{L_O} = \iiint\limits_{\Sigma} \overrightarrow{OM} \wedge \overrightarrow{v}(M,t)\mu(M,t)\mathrm{d}\tau$

For a *closed system* Σ^*, the **Angular Momemtum Theorem** reads $\boxed{\dfrac{\mathrm{d}\overrightarrow{L_O}^*}{\mathrm{d}t} = \sum_i \overrightarrow{M_O}(\overrightarrow{F_i})}$

In practice we always use the local equation between t and $t + \mathrm{d}t$.

13.6 Fluid Dynamics

Consider a speed field $\vec{v}(x,y,z,t)$. Its differential is $\mathrm{d}\vec{v} = \dfrac{\partial \vec{v}}{\partial t}\mathrm{d}t + \dfrac{\partial \vec{v}}{\partial x}\mathrm{d}x + \dfrac{\partial \vec{v}}{\partial y}\mathrm{d}y + \dfrac{\partial \vec{v}}{\partial z}\mathrm{d}z$.

During a time interval $\mathrm{d}t$, a fluid particle travels a distance $\overrightarrow{\mathrm{d}r} = \vec{v}\mathrm{d}t$: we get $\begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \\ \mathrm{d}z \end{pmatrix} = \begin{pmatrix} v_x\mathrm{d}t \\ v_y\mathrm{d}t \\ v_z\mathrm{d}t \end{pmatrix}$.

Therefore $\mathrm{d}\vec{v} = \dfrac{\partial \vec{v}}{\partial t}\mathrm{d}t + \dfrac{\partial \vec{v}}{\partial x}v_x\mathrm{d}t + \dfrac{\partial \vec{v}}{\partial y}v_y\mathrm{d}t + \dfrac{\partial \vec{v}}{\partial z}v_z\mathrm{d}t = \dfrac{\partial \vec{v}}{\partial t}\mathrm{d}t + (\vec{v}.\overrightarrow{\mathrm{grad}})\vec{v}\mathrm{d}t$.

So $\vec{a} = \dfrac{1}{\mathrm{d}t} \times \mathrm{d}\vec{v} = \dfrac{\partial \vec{v}}{\partial t} + (\vec{v}.\overrightarrow{\mathrm{grad}})\vec{v}$. To avoid confusion we note $\vec{a} = \dfrac{D\vec{v}}{Dt}$

Acceleration

The acceleration is $\boxed{\vec{a} = \dfrac{D\vec{v}}{Dt} = \dfrac{\partial \vec{v}}{\partial t} + (\vec{v}.\overrightarrow{\mathrm{grad}})\vec{v}}$ $(\vec{v}.\overrightarrow{\mathrm{grad}})\vec{v}$ is the "convective derivative".

Vorticity

The **vorticity** is the vector $\boxed{\vec{\Omega} = \dfrac{1}{2}\overrightarrow{\mathrm{rot}}\,\vec{v}}$.

For a incompressible flow, $\mathrm{div}\vec{v} = \vec{0}$ and $\boxed{\oint \vec{v}.\vec{\mathrm{d}l} = 2\iint \vec{\Omega}.\vec{\mathrm{d}S}}$ (Ampere's theorem)

A flow is said to be **irrotational** when $\vec{\Omega} = \vec{0}$

For an irrotational flow, $\overrightarrow{\mathrm{rot}}\,\vec{v} = \vec{0}$ so $\boxed{\vec{v} = \overrightarrow{\mathrm{grad}}\phi}$ where ϕ is the **speed potential**

We therefore have Laplace's equation $\Delta\phi = 0$

General solutions of the Laplace equation for $\phi = \phi(r,\theta)$ are :

$$\phi(r,\theta) = \alpha_0 \ln(r) + \beta_0 + \sum_{n=1}^{+\infty}(\alpha_n r^n + \beta_n r^{-n})\cos(n\theta) + \sum_{n=1}^{+\infty}(\gamma_n r^n + \delta_n r^{-n})\sin(n\theta)$$

Example : flow around an infinite cylinder.

For the next theorem we use the formula $(\vec{f}.\overrightarrow{\mathrm{grad}})\vec{f} = \overrightarrow{\mathrm{grad}}\dfrac{f^2}{2} + \overrightarrow{\mathrm{rot}}\,\vec{f} \wedge \vec{f}$

Euler's equation

The volumic PFD yields $\boxed{\mu\dfrac{\partial \vec{v}}{\partial t} + \mu\overrightarrow{\mathrm{grad}}\dfrac{v^2}{2} + \mu\overrightarrow{\mathrm{rot}}\,\vec{v} \wedge \vec{v} = -\overrightarrow{\mathrm{grad}}P + \vec{\varphi}}$

Where $\vec{\varphi}$ is the resultant of the non-pressure volumic forces.

Example : oscillations of a fluid in a U-tube.

Bernouilli's equation

For an incompressible, homogenous, stationary, ideal *and irrotational* flow :

$\boxed{P + \mu g z + \frac{1}{2}\mu v^2 = K}$ (K is independent of time and of position)

Examples :

- MAGNUS effect : a ball with top spin falls faster.
- Lift force on an airplane wing.
- Vortexes in the wake of airplanes.

Navier-Stokes equation

The volumic PFD for a viscous fluid yields $\boxed{\mu\frac{\partial \vec{v}}{\partial t} + \mu(\vec{v}.\overrightarrow{\text{grad}})\vec{v} = -\overrightarrow{\text{grad}}P + \mu\vec{g} + \eta\vec{\Delta}\vec{v}}$

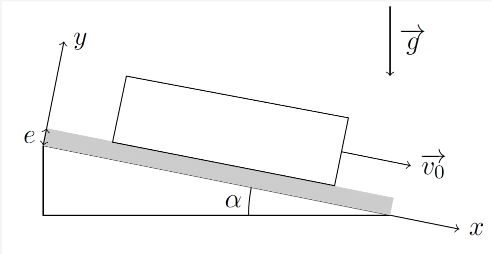
13.7 TD

Ex 50

We consider an object of mass m sliding on oil at a constant speed $\vec{v_0}$.

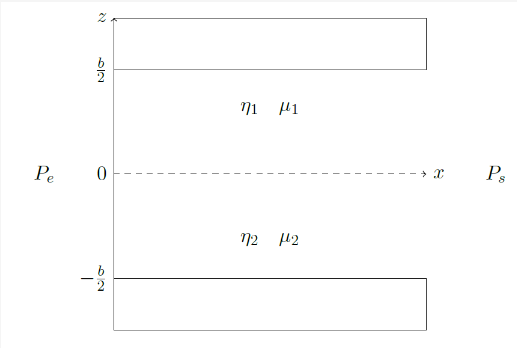
1) Compute the speed field $\vec{v}$ inside the oil. You may make approximations in order to simplify the computations.

2) Find the value of $\vec{v_0}$.



Ex 51

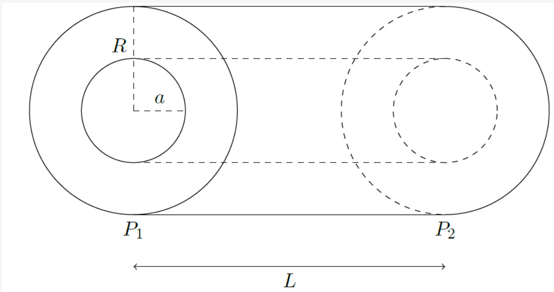
Two non-miscible fluids are disposed one under the other in a pipe, with a pressure difference $P_e - P_s > 0$.



Determine the speed field inside the fluid.

Ex 52

We consider a two cylinders inside each other with a common axis z. The plane $z = 0$ has a uniform pressure P_1 and the plane $z = L$ has a uniform pressure P_2. We suppose that the speed field is of the form $v(r)\vec{u_z}$ in both cylinders.



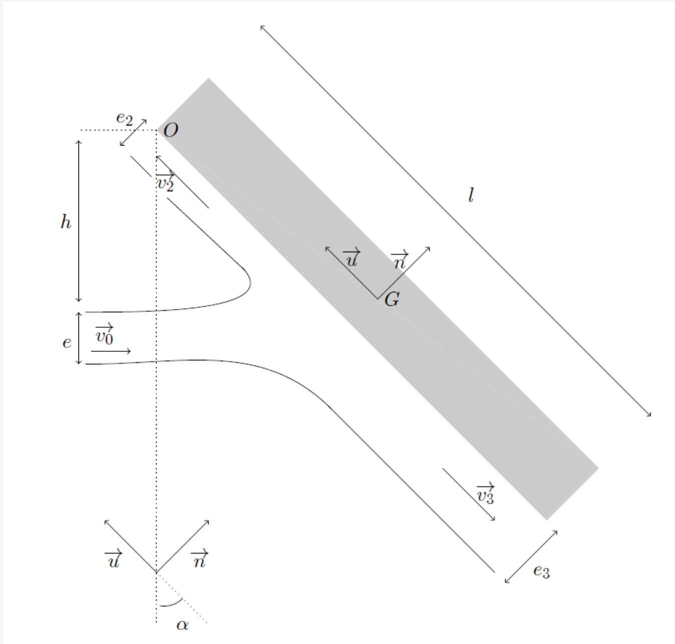
1) Establish that at every point, $\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}v}{\mathrm{d}r}\right) = \frac{P_2 - P_1}{\eta L}r$

2) Determine $\vec{v}$ for $a \leq r \leq R$.

3) Find the hydraulic resistance R_2 of the part $a \leq r \leq R$ and remind the formula of R_1 (portion $r \leq a$). For $L = 1m, a = 2cm, R = 4cm$, give the values of R_1 and R_2. Can the resistance of a pipe of radius R and length L be considered as an association of R_1 and R_2 in parallel ? Compare numerical values.

Ex 53

We consider a plate held by its top part (axis Δ passing by O) that is sprayed by a beam of water at a speed $\overrightarrow{v_0}$ and width e. This spraying pushes it to an equilibrium angle α with the vertical. It has a length l and width L (L is perpendicular to the plane of the drawing). The spray of water splits into two : one going upwards of width e_2 and speed $\overrightarrow{v_2}$ and one going downwards of width e_3 and speed $\overrightarrow{v_3}$.



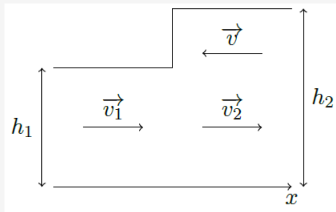
1) Associate a closed system to the flowing water and the plate together and use the angular momentum theorem in order to determine the equilibrium angle α.

2) Determine $\overrightarrow{v_2}, \overrightarrow{v_3}, e_2$ and e_3 as a function of the other parameters.

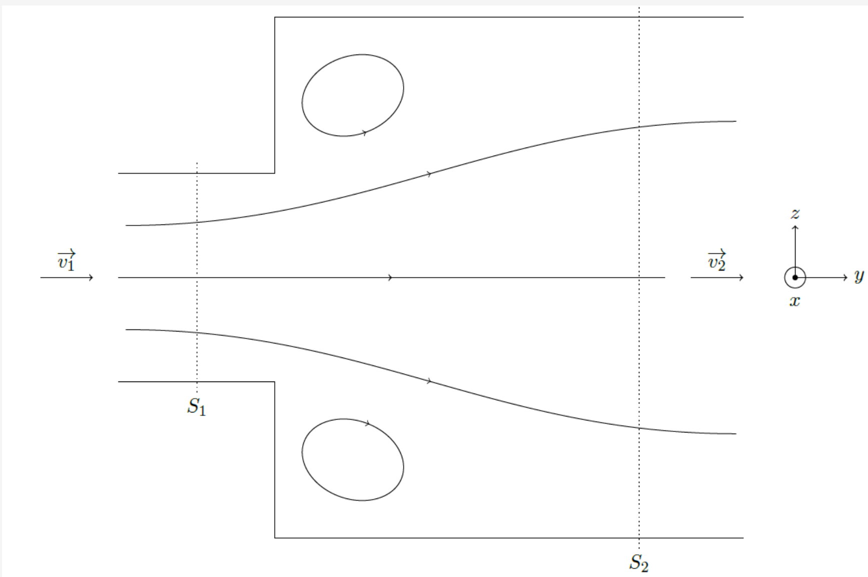
Ex 54

We consider a wave going upstream a river at a constant speed $\overrightarrow{v}$. The speed of the river before the wave is $\overrightarrow{v_1}$, and $\overrightarrow{v_2}$ after the wave. We consider the flow to be ideal and incompressible.

Compute $\overrightarrow{v}$ and $\overrightarrow{v_2}$ depending on the other parameters.



Ex 55



We consider an increase in section for a turbulent, homogenous and incompressible flow. Compute the singular head loss.

PART III

Tests

1 Maths December Test

1

Let $\theta \in \mathbb{R}$.

a) Expand $\cos(4\theta)$ into a polynomial in $\cos\theta$

We now consider $\theta = \dfrac{\pi}{5}$ until the end of the exercise.

b) Let $c = \cos\theta$. Prove that $8c^4 - 8c^2 + c + 1 = 0$.

c) We give $8X^4 - 8X^2 + X + 1 = (X+1)(X - \frac{1}{2})(8X^2 - 4X - 2)$. Compute $\cos\frac{\pi}{5}$.

2

a) Prove that $\forall x \in [-1, 1], \quad \cos(\text{Arcsin}(x)) = \sqrt{1-x^2}$.

b) Using question **2a)**, solve by *Analysis/Synthesis* the equation

$$\text{Arcsin}\left(\frac{\sqrt{x}}{2}\right) + \text{Arcsin}\left(\frac{1}{\sqrt{x}}\right) = \frac{\pi}{2} \qquad \text{for } x \in [1, 4]$$

3

Let $n \in \mathbb{N}$. Solve by equivalence the equation $(z+1)^n = (z-1)^n$ for $z \in \mathbb{C}$.
Express the solutions in the most simple way possible.

Bonus

1) We note $F(\mathbb{R}, \mathbb{R})$ the set of the functions from $\mathbb{R}$ to $\mathbb{R}$. Let $f \in F(\mathbb{R}, \mathbb{R})$.

We define $L_f : \begin{cases} F(\mathbb{R}, \mathbb{R}) & \longrightarrow & F(\mathbb{R}, \mathbb{R}) \\ g & \longmapsto & f \circ g \end{cases}$

Reminder : for $g \in F(\mathbb{R}, \mathbb{R})$, we have $\forall x \in \mathbb{R}, \quad (f \circ g)(x) = f(g(x))$

a) On what condition on f is L_f injective ?

b) On what condition on f is L_f surjective ?

2) Let E be a set. We define $\mathcal{P}(E) = \{A | A \subset E\}$ (the set of all the subsets of E.)

Does there exist an injection $f : E \longrightarrow \mathcal{P}(E)$?

Does there exist a surjection $f : E \longrightarrow \mathcal{P}(E)$?

2 Physics December Test

2.1 The effect of friction on an orbit

We consider a satellite of mass m at a circular orbit of radius r around the Earth (of mass M_E).

1) Determine the potential energy E_p of the satellite.

2) Show that its speed is $v = \sqrt{\dfrac{GM_E}{r}}$.

3) Calculate its kinetic energy E_c and compare it to E_p.

4) Express its mechanical energy E_m.

We now consider that the satellite is affected by a friction force $\overrightarrow{f} = -\alpha m v \overrightarrow{v}$, in addition to the gravitation force $\overrightarrow{F_G}$. The orbit is now slightly elliptical.

5) What is the dimension of α ?

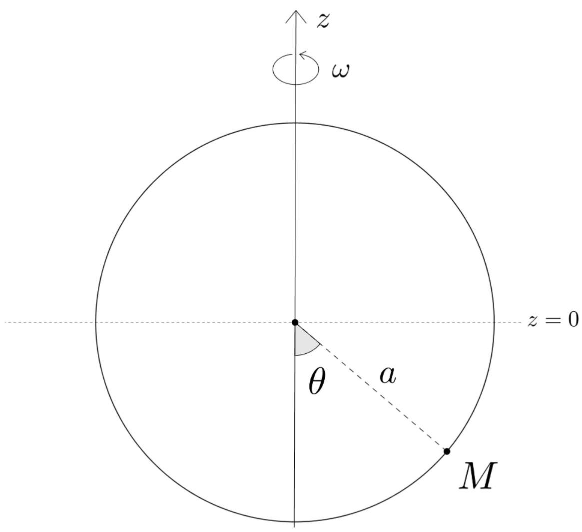
6) Prove that $P(\overrightarrow{F_G}) = -\dfrac{\mathrm{d}E_p}{\mathrm{d}t}$.

7) Using the Kinetic Energy Theorem, prove that $\dfrac{\mathrm{d}E_m}{\mathrm{d}t} = \overrightarrow{f}.\overrightarrow{v}$.

8) Considering the value of E_m found in **4)** to be true and the value of v found in **2)** to be true, find a differential equation on r.

9) Is the satellite falling towards the Earth ? How does its speed v vary ?

2.2 A rotating circle



A circle of radius a is rotating around the z axis at a constant angular speed ω. We study the movement of a ring M of mass m sliding on the circle (it moves along it without friction).

1) Calculate the potential energy $E_{pd}(\theta)$ attached to the drive force (*force d'inertie d'entraînement*).

2) Calculate the potential energy $E_{pw}(\theta)$ attached to the weight.

3) Calculate the kinetic energy $E_c(\theta)$ of the ring in the rotating referential.

4) Explain why we have the conservation of the mechanical energy. Differentiate that equation according to time and prove : $\ddot{\theta} = -\omega_0^2 \sin\theta + \omega^2 \sin\theta\cos\theta$, where $\omega_0^2 = \frac{g}{a}$.

This re-writes into $\boxed{(E) : \ddot{\theta} = \omega_0^2(\lambda\cos\theta - 1)\sin\theta}$ (with $\lambda = \frac{\omega^2}{\omega_0^2}$.)

5) We suppose $\lambda < 1$. Using (E), find the two equilibrium points $\theta_{eq1}, \theta_{eq2}$. Discuss their stability without any calculations.

6) We now suppose $\lambda \geq 1$. Using (E) find two **other** opposed equilibrium points $\pm\theta_0$.

7) Let $\varepsilon = \theta - \theta_0$. Considering ε to be very small, find a linear differential equation on ε and discuss the stability of θ_0.

3 Liner Filtration TP

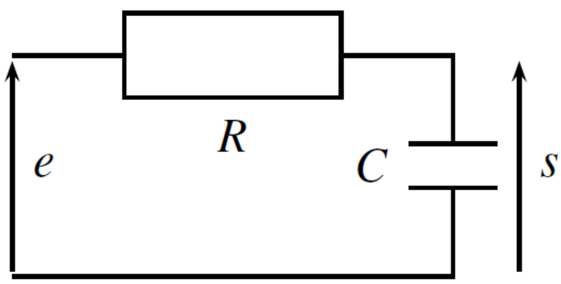
Objectives of the TP :

- Study experimentally the filters seen in the Electricity lesson.
- Determine experimentally the nature of unknown filters.
- You must hand in a TP report at the end of the session. We recommend having a student per team responsible of writing down the answers to the questions.

For all circuits, take $R = 100\Omega$, $L = 0,2H$ and $C = 0,47\mu F$.

3.1 Experimental study of the lesson filters

3.1.1 Pass-low of the first order

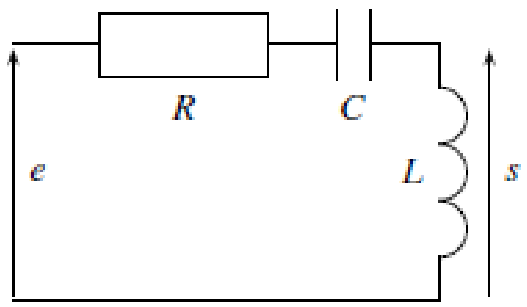


1) Anticipate the behaviour of the filter by giving the transfer function, give the theoretical value of the self-frequency f_0.

2) Build the circuit with $R = 100\Omega$ and $C = 0,47\mu F$

3) Check experimentally the behaviour of the filter and the self-frequency, give numerical values to explain your results. Use the "sweep" function to quickly find the nature of the filter.

3.1.2 Pass-high of the second order

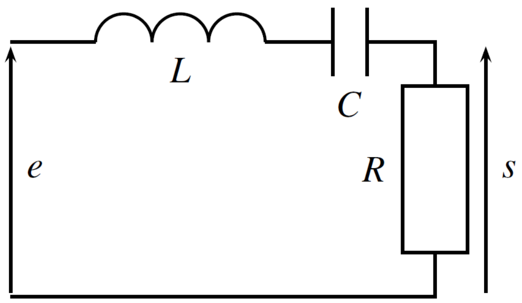


1) Anticipate the behaviour of the filter by giving the transfer function, give the theoretical value of the self-frequency f_0 and of the quality factor Q. Is there resonance for $R = 100\Omega$, $L = 0,2H$ and $C = 0,47\mu F$?

2) Build the circuit.

3) Check experimentally the behaviour of the filter (resonance, self-frequency, behaviour at the extremes...).

3.1.3 Band-Pass



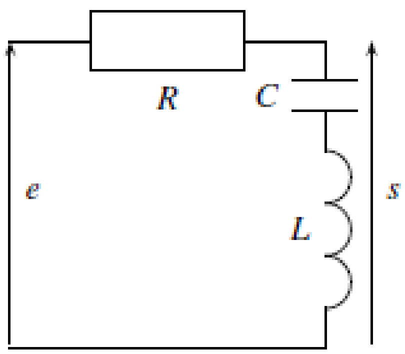
1) Anticipate the behaviour of the filter by giving the transfer function, give the theoretical value of the self-frequency f_0.

2) Build the circuit.

3) Check experimentally the behaviour of the filter (self-frequency, behaviour at the extremes...).

3.2 Study of new filters

3.2.1 First unknown filter

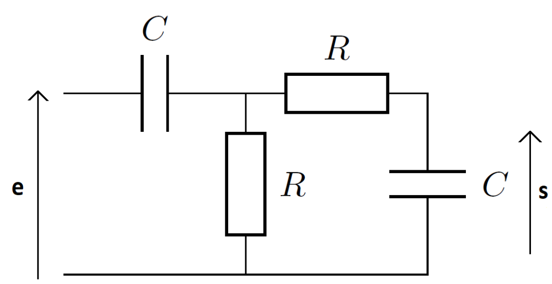


1) Anticipate the behaviour of the filter by analysing its transfer function. What type of filter is this ? What use does it have ?

2) Build the circuit.

3) Check experimentally the behaviour of the filter.

3.2.2 Second unknow filter



Study experimentally the filter. What type of filter is it ?