A Fixed-Point Algorithm for Computing Robust Barycentres of Measures

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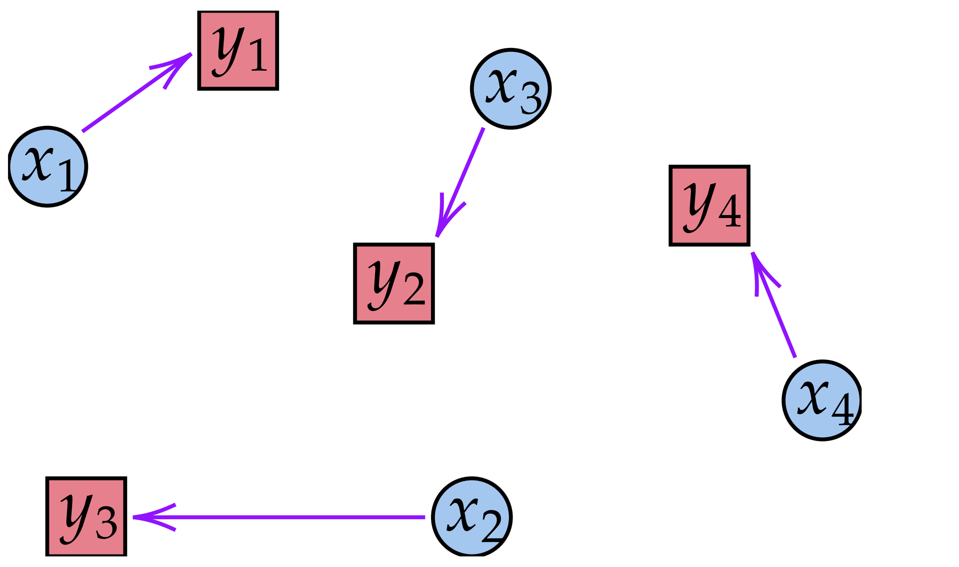
1.1 The Monge Problem



$\mu := \frac{1}{n}\sum_{i=1}^{n}\delta_{x_i}$, $\nu := \frac{1}{n}\sum_{i=1}^{n}\delta_{y_i}$

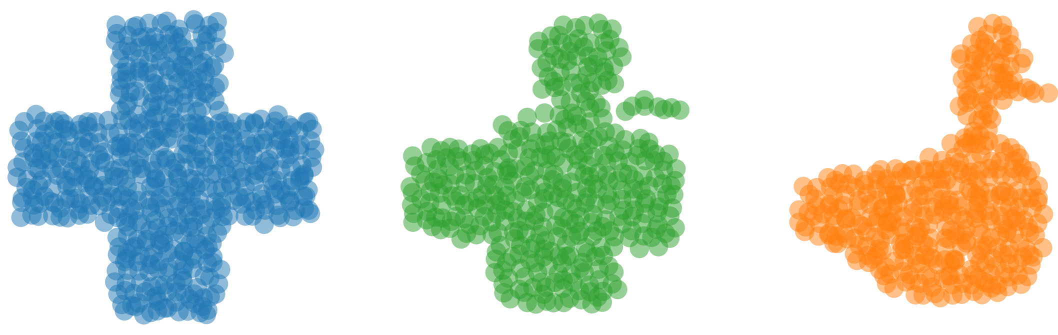
$\mathcal{T}_c(\mu,\nu) := \min_{\sigma\in\mathfrak{S}_n} \frac{1}{n}\sum_{i=1}^{n} c(x_i, y_{\sigma(i)})$.

- p, q cost: $c_{p,q}(x,y) := \|x-y\|_p^q$.
- q-Wasserstein: $W_q^q := \mathcal{T}_{c_{p=2,q=q}}$.

1.2 2-Wasserstein Barycentres

Euclidean Fréchet mean: $\underset{x\in\mathbb{R}^d}{\text{argmin}} \sum_{k=1}^{K}\lambda_k\|x-y_k\|_2^2 = \sum_{k=1}^{K}\lambda_k y_k$.

W_2^2 barycentre (Agueh and Carlier 2011): $\underset{\mu\in\mathcal{P}(\mathbb{R}^d)}{\text{argmin}} \sum_{k=1}^{K}\lambda_k W_2^2(\mu,\nu_k)$.



1.3 Fixed-Point Method for W_2^2 barycentres

- Discrete case: Cuturi and Doucet 2014.
- Proofs for absolutely continuous case: Álvarez-Esteban et al. 2016.

Objective: $\underset{(x_i)_{i=1}^n}{\text{argmin}} \sum_{k=1}^{K}\lambda_k \min_{\sigma_k\in\mathfrak{S}_n}\sum_{i=1}^{n}\|x_i - y_{k,\sigma_k(i)}\|_2^2 = \sum_{k=1}^{K}\lambda_k W_2^2(\mu,\nu_k)$.

Fixed-Point Algorithm for W_2^2 Barycentres

Input: For $k\in[\![1,K]\!]$, target $(y_{k,i})_{i=1}^n$. Weights (λ_k).

for $t\in[\![0,T-1]\!]$ **do**

For $k\in[\![1,K]\!]$, $\sigma_k \in \underset{\sigma\in\mathfrak{S}_n}{\text{argmin}} \sum_{i=1}^{n}\|x_i^{(t)} - y_{k,\sigma(i)}\|_2^2$;

For $i\in[\![1,n]\!]$, $x_i^{(t+1)} = \sum_{k=1}^{K}\lambda_k y_{k,\sigma_k(i)}$;

2.1 Barycentres for Generic Transport Costs

Objective: $\underset{(x_i)_{i=1}^n}{\text{argmin}} \sum_{k=1}^{K}\lambda_k \min_{\sigma_k\in\mathfrak{S}_n}\sum_{i=1}^{n} c_k(x_i, y_{k,\sigma_k(i)}) = \sum_{k=1}^{K}\lambda_k \mathcal{T}_{c_k}(\mu,\nu_k)$.

Fixed-Point Algorithm for OT Barycentres

Input: For $k\in[\![1,K]\!]$, target $(y_{k,i})_{i=1}^n$. Weights (λ_k).

for $t\in[\![0,T-1]\!]$ **do**

For $k\in[\![1,K]\!]$, $\sigma_k \in \underset{\sigma\in\mathfrak{S}_n}{\text{argmin}} \sum_{i=1}^{n} c_k(x_i^{(t)}, y_{k,\sigma(i)})$;

For $i\in[\![1,n]\!]$, $x_i^{(t+1)} = \underset{x}{\text{argmin}} \sum_{k=1}^{K}\lambda_k c_k(x, y_{k,\sigma_k(i)})$;

Long paper: Tanguy et al. 2024 for formulation with generic measures.

2.2 Algorithm Convergence

- Assuming only continuous costs and measures on compact metric spaces.
- Extends Álvarez-Esteban et al. 2016.

Energy function: $V(\mu) := \sum_{k=1}^{K}\lambda_k W_2^2(\mu,\nu_k)$. Iterates: $\mu_{t+1}\in G(\mu_t)$.

Decrease Property

$$\forall \mu \in G(\mu),\ V(\mu) \geq V(\mu) + \mathcal{T}_\delta(\mu,\mu).$$

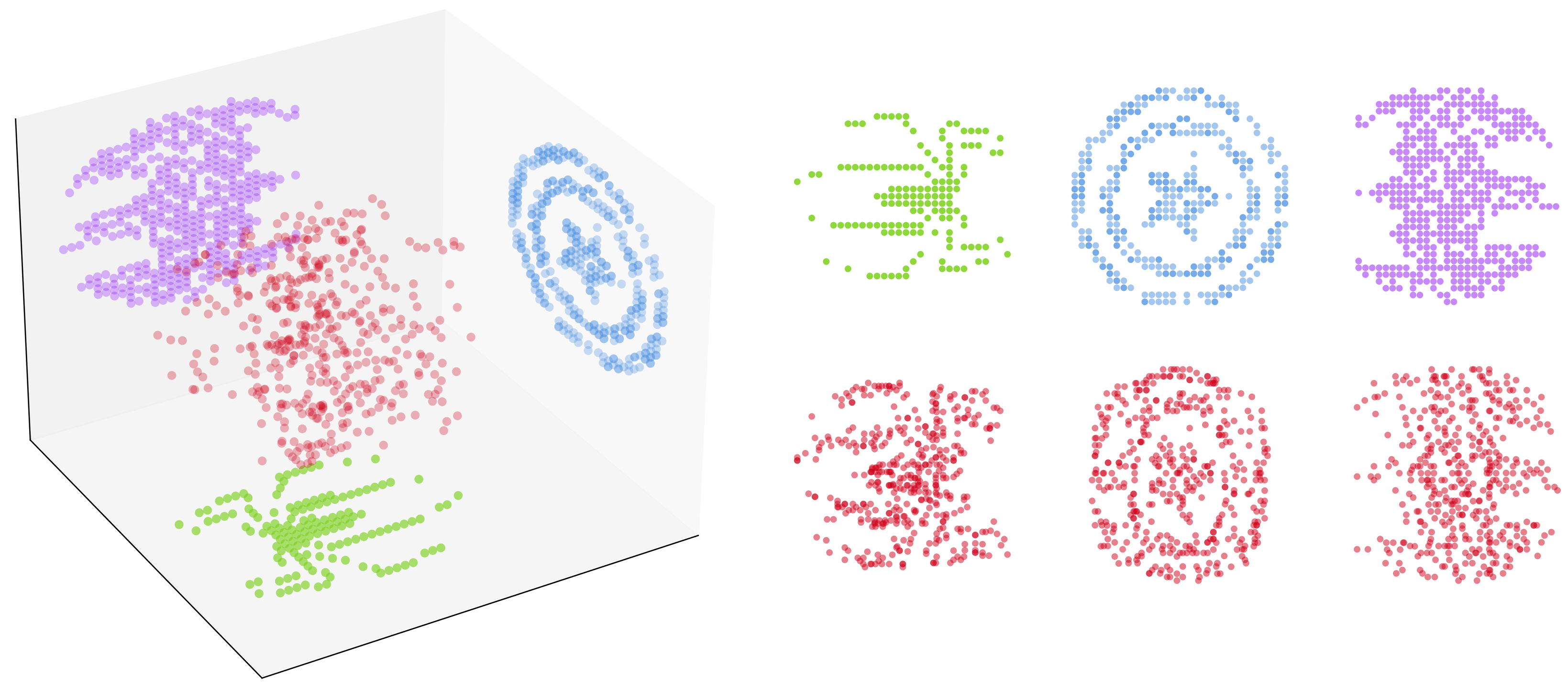
If μ^* is a barycentre then $G(\mu^*) = \{\mu^*\}$.

Convergence

If μ is a subsequential limit of (μ_t) then $\mu\in G(\mu)$.

3.1 3D Barycentre of 2D Point Clouds

Objective: $\underset{(x_i)_{i=1}^n\in\mathbb{R}^{n\times 3}}{\text{argmin}} \frac{1}{K}\sum_{k=1}^{K}\min_{\sigma_k\in\mathfrak{S}_n}\sum_{i=1}^{n}\|P_k x_i - y_{k,\sigma_k(i)}\|_2$.

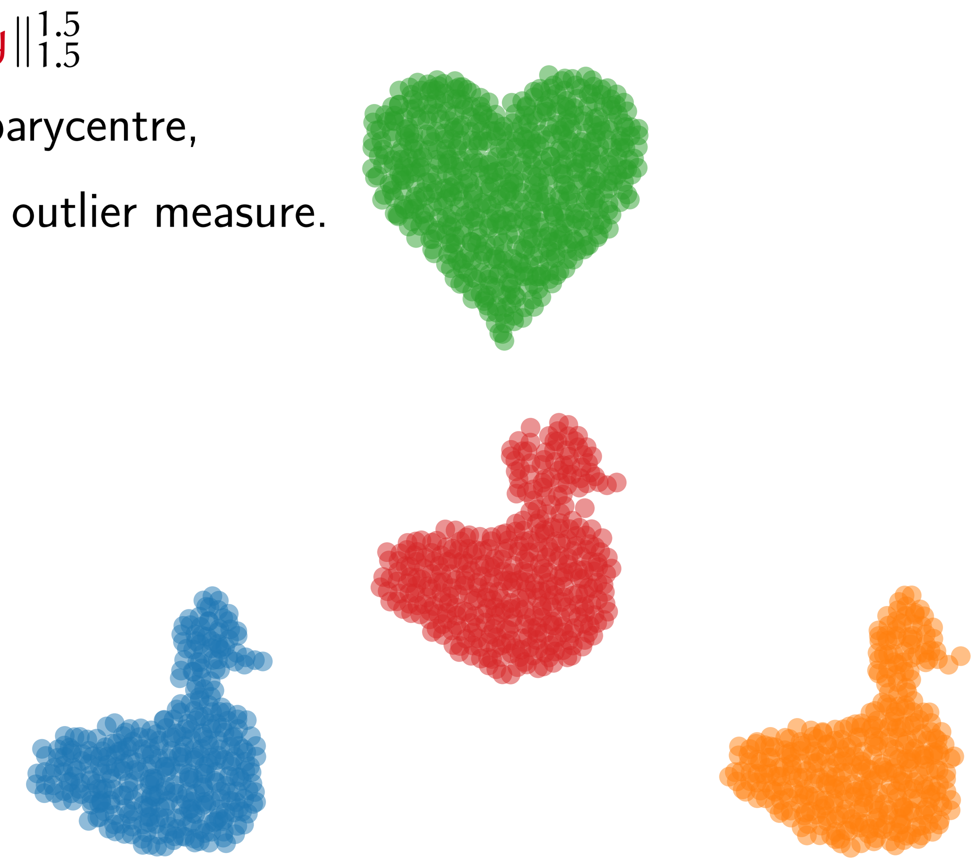


3.2 Robust Barycentres

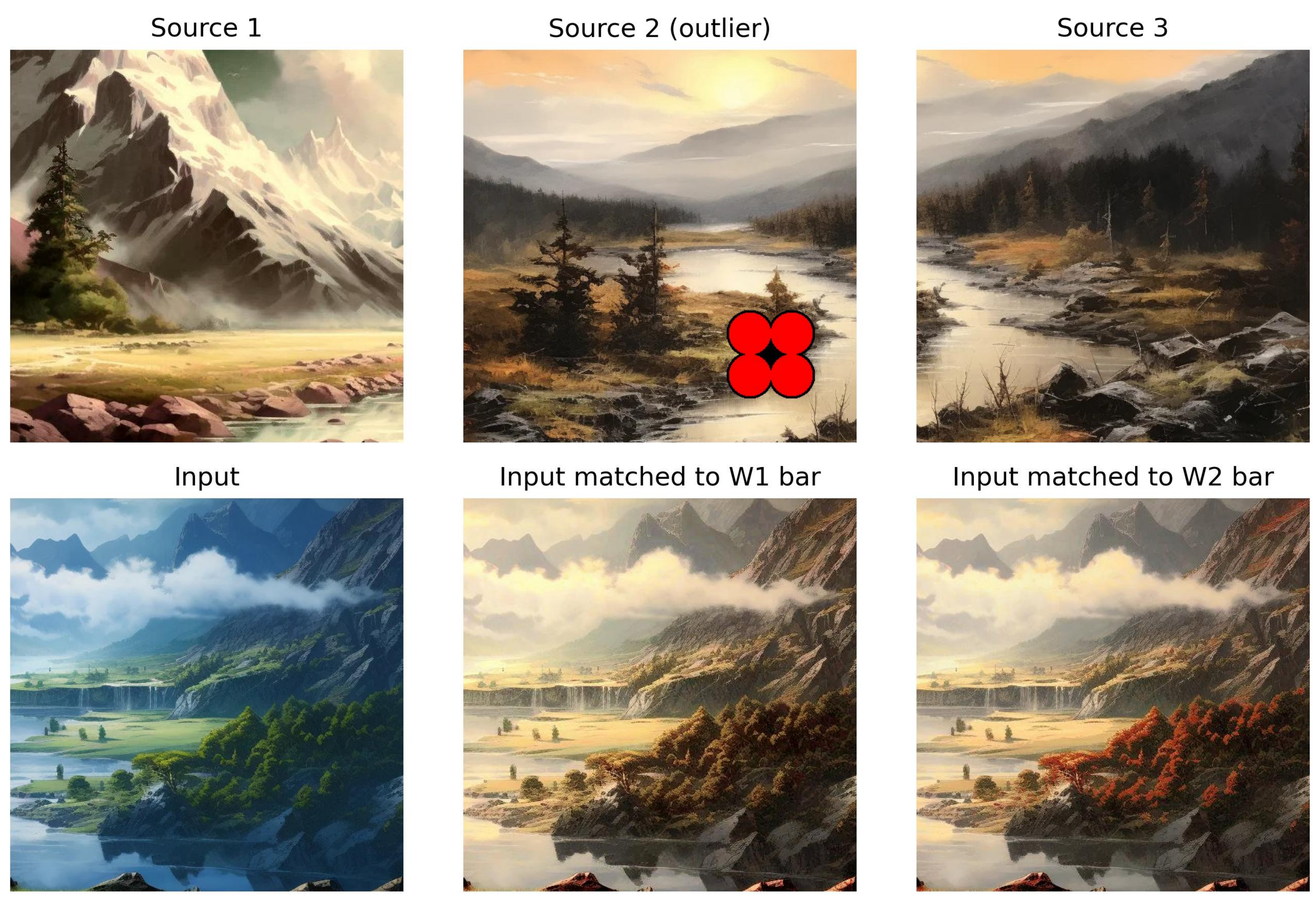
$c(x,y) := \|x-y\|_{1.5}^{1.5}$

1.5-Wasserstein barycentre,

Robust to (heart) outlier measure.



3.3 Colour Transfer With Outliers



[1] Martial Agueh and Guillaume Carlier. “Barycenters in the Wasserstein Space”. In: *SIAM Journal on Mathematical Analysis* 43.2 (2011), pp. 904–924.

[2] Pedro C Álvarez-Esteban, E Del Barrio, JA Cuesta-Albertos, and C Matrán. “A fixed-point approach to barycenters in Wasserstein space”. In: *Journal of Mathematical Analysis and Applications* 441.2 (2016), pp. 744–762.

[3] Marco Cuturi and Arnaud Doucet. “Fast Computation of Wasserstein Barycenters”. In: *Proceedings of the 31st International Conference on Machine Learning*. Vol. 32. Proceedings of Machine Learning Research. Bejing, China: PMLR, June 2014, pp. 685–693.

[4] Eloi Tanguy, Julie Delon, and Nathaël Gozlan. *Computing Barycentres of Measures for Generic Transport Costs*. 2024. arXiv: 2501.04016 [math.NA].